APMTH-105 Midterm 3 Review

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Problem 1: Diffusion Equation with Delta Functions BC, using Greens Function

From: Practice Midterms

Consider the diffusion equation

$$\partial_t u = D\partial_{xx}, u$$

with $-\infty \leq x \leq \infty$, with the initial condition $u(x, 0) = 10\delta(x - 10) + 4\delta(x) + 3\delta(x + 10)$.

Solution:

Using a Green's function, the solution is given by

$$u(x, t) = \int_{-\infty}^{\infty} G(x - x', t)u(x', 0)\; dx'.$$

Using the initial condition given, we thus have that

$$u(x, t) = 10G(x - 10, t) + 4G(x, t) + 3G(x + 10, t).$$

Using

$$G(x - x', t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-x')^2/(4Dt)},$$

we can evaluate the different terms in the solutions at the different times given, where $\sqrt{4Dt} = 2\sqrt{2}, 4, 8$ respectively. What I am most concerned about is that you compute the maximum heights and the widths. For example at $t = 4$, the first peak $10G(x - 10, t)$ has a height $10/\sqrt{4\pi 4} = 5/2/\sqrt{\pi}$.The width is$\sqrt{4Dt} = \sqrt{16} = 4$. If you sketch these heights and widths for all of the solutions you will see how they evolve.

Problem 2: Diffusion Advection and Forcing Equation

From: Practice Midterms

Consider the equation

$$\partial_t n = D\partial_{xx} n - 3\partial_x n + Hn,$$

on the interval $0 \leq x \leq L$, with the boundary conditions $u(0, t) = u_1$ and $u(L, t) = u_2$. Suppose the initial condition is $u(x, 0) = f(x)$. Find the solution $u(x, t)$.

Solution:

We note first that this is an inhomogeneous problem: the boundary conditions are inhomogeneous.Thus we expect the solution to be of the form $n = n_s + n_d$, where n_s is the steady solution and n_d solves the Dirichlet problem. To find n_s we note it obeys

$$D\partial_{xx} n_s - 3\partial_x n_s + Hn_s = 0.$$

This is a second order ODE, and we can find its solution by the ansatz $u_s = e^{\beta x}$, which gives that

$$D\beta^2 - 3\beta + H = 0,$$

or

$$\beta_\pm = \frac{3 \pm \sqrt{9-4HD}}{2D}.$$

Depending on the sign of the discrimenant these are either real or imaginary. But regardless the solution for n_s is

$$n_s = Ae^{\beta_+ x} + Be^{\beta_- x}.$$

Now we apply the boundary conditions $n_s(0) = A + B = u_1$ and $n_s(L) = Ae^{\beta_+ L} + Be^{\beta_- L} = u_2$. This is two equations and 2 unknowns and therefore can be solved.

To find n_d we proceed as always, with separation of variables. We write $n_d = X(x)T(t)$. The differential equation for X is then

$$DX'' - 3X' + (H + \kappa^2)X = 0,$$

and that for T is

$$\dot{T} = -\kappa^2 T.$$

Hence we have $T = T_0 e^{-\kappa^2 t}$, whereas for $X(x)$ we again make the ansatz that $X(x) = e^{\alpha x}$, which means that α obeys

$$D\alpha^2 - 3\alpha + (H + \kappa^2) = 0.$$

This gives

$$\alpha_\pm = \frac{3 \pm \sqrt{9 - 4D(H+\kappa^2)}}{2D}.$$

Now we have spent lots of time in the past messing with the different cases for the discrimenant. Here lets just be realists and realize that the boundary conditions can only be satisfied when α is imaginary–ie $4(H+\kappa^2) - 9 > 0$. We can then write the general solution as

$$X(x) = e^{\frac{3x}{2D}} \left(A \sin\left(\frac{\sqrt{4D(H+\kappa^2)-9}}{2D} x \right) + B \cos\left(\frac{\sqrt{4D(H+\kappa^2)-9}}{2D} x \right) \right).$$

We apply boundary conditions $X(0) = X(L) = 0$, which implies that $B = 0$ and that

$$\frac{\sqrt{4D(H+\kappa^2)-9}}{2D} L = n\pi.$$

There are therefore a countable set of solutions, for κ_n and the most general solution is of the form

$$n_d(x,t) = \sum_{n=1}^{\infty} e^{-\kappa_n^2 t} A_n e^{\frac{3x}{2D}} \sin\left(\frac{n\pi}{L} x\right).$$

We can therefore apply the initial conditions $n_d(x,0) = f(x) - n_s(x)$, and use orthogonality to find the A_n's. Note that applying the orthogonality relationship requires putting our ODE for X into Sturm Liouville form, and finding the weighting function. As always we can do this by multiplying by $\sigma(x)$ to get

$$D\sigma X'' - 3\sigma X' + (H+\kappa^2)\sigma X = 0 \Rightarrow \sigma X'' - \frac{3}{D}\sigma X' + \frac{(H+\kappa^2)}{D}\sigma X = 0.$$

If we choose $\sigma = e^{\frac{-3x}{D}}$ then we can rewrite this equation as

$$\frac{d}{dx}\left(e^{\frac{-3x}{D}} DX' \right) + (H+\kappa^2) e^{\frac{-3x}{D}} X = 0,$$

so that the weighting function $w(x) = e^{\frac{-3x}{D}}$.

Problem 3: Laplacian Operator Problem

From: Practice Midterms

Find $u(x, y)$ satisfying $\nabla^2 u = 1$ for $0 \leq x \leq a$ and $0 \leq y \leq b$ with the boundary conditions:
i. $u(x, 0) = 0$
ii. $\partial_y u(x, b) \equiv u_y(x, b) = 0$
iii. $u(0, y) = 0$
iv. $u(a, y) = 0$

Solution:

Here is the way to parse it: First, lets find the eigenfunctions of the laplacian operator that satisfy the boundary conditions. I.e, we should solve

$$\nabla^2 u = -\lambda^2 u.$$

To solve this we can just use separation of variables–namely, write $u(x, y) = X(x)Y(y)$. We can then get the two ODE's $X'' + \alpha^2 X = 0$, and $Y'' + \beta^2 Y = 0$, where necessarily $\alpha^2 + \beta^2 = \lambda^2$. We now need both $X(x)$ and $Y(y)$ to satisfy the boundary conditions. For $X(x)$, this means that $X(0) = X(a) = 0$. As we have done many times in these solutions already, this implies that $X(x) = A\sin(\alpha x) + B\cos(\alpha x)$, so that $B = 0$ and $\alpha a = n\pi$. Similarly for $Y(y) = C\sin(\beta y) + D\cos(\beta y)$, we have that $Y(0) = D = 0$, and $Y'(b) = \beta C\cos(\beta b) = 0$, which implies that $\beta b = (n + \frac{1}{2})\pi$.

Putting this together we have that $\lambda^2_{n,m} = \alpha_n^2 + \beta_m^2 = (n\pi/a)^2 + (m + \frac{1}{2})^2\pi^2/b^2$.

Why is this useful? Now let us expand the function 1 in terms of the eigenfunctions we have just constructed. Namely we'd like to write

$$1 = \sum_{mn} A_{nm}\sin(\alpha_n x)\sin(\beta_m y).$$

We can now use the orthogonality of the eigenfunctions to solve for the A_{nm}'s, i.e

$$A_{nm} = \frac{\int_0^a dx\ \int_0^b\ dy\sin(\alpha_n x)\sin(\beta_m y)}{\int_0^a dx\ \int_0^b\ dy\sin^2(\alpha_n x)\sin^2(\beta_m y).}$$

This means we can take the right hand side and expand it in our original equation. Namely, lets write

$$\nabla^2 u = 1 = \sum_{mn} A_{nm}\sin(\alpha_n x)\sin(\beta_m y).$$

If we then just plug in that $u = \sum_{nm} U_{nm}\sin(\alpha_n x)\sin(\beta_m y)$, we can immediately solve for the U_{nm}, namely

$$U_{nm} = \frac{A_{nm}}{\alpha_n^2 + \beta_m^2}.$$

This is the solution to the problem–you should see though that this is much too hard for the midterm, and even goes too far afield to be a good problem for the final. However you should see that the logical structure that we have built allows us to address it!

Problem 4: Method of Characteristics

From: Practice Midterms

Solve the following equations using the method of characteristics. Assume in each case that the initial condition is $u(x,0) = e^x \sin(x)$.

(a)

$$\partial_t u + 6\partial_x u = 5 + \frac{6}{t+1}.$$

(b)

$$\partial_t u + \partial_x u = e^u.$$

(c)

$$\partial_t u + x\partial_x u + u = 1.$$

Solution: (a)

Recalling the formalism from class and from the characteristics solutions, we have $dx/dt = a = 6$, so that $x(t) = 6t + x_0$. We also have $du/dt = 5 + 6/(t+1)$, so that $u(t) = 5t + 6\log(t+1) + A$. Now, if we apply initial conditions, we have $u(t=0) = A = u(x,0) = e^{x_0}\sin(x_0)$. If we now use that $x_0 = x - 6t$, this gives

$$\boxed{u(x,t) = 5t + 6\log(t+1) + e^{x-6t}\sin(x-6t).}$$

Solution: (b)

Here we have $dx/dt = 1$ and $du/dt = e^u$. This gives $x(t) = x_0 + t$ and $u(t) = -\log(C-t)$. Hence the initial conditions are that $u(t=0) = -\log(C)$. Using our initial conditions, this gives that $-\log(C) = e^{x_0}\sin(x_0)$, or $C = e^{-e^{x_0}\sin(x_0)}$. Hence putting it all together implies that

$$\boxed{u(x,t) = -\log(e^{e^{x-t}\sin(x-t)} - t).}$$

Solution: (c)

OK. Here we have $dx/dt = x$ and $du/dt = 1-u$. These give $x(t) = x_0 e^t$ and $u(t) = 1 - Ae^{-t}$. Initial conditions imply that $1 - A = e^{x_0}\sin(x_0)$. Putting it all together implies that $x_0 = xe^{-t}$, so that

$$\boxed{u(x,t) = 1 - (1 - e^{xe^{-t}}\sin(xe^{-t}))e^{-t}.}$$

Note that one thing that you need to be careful about with these solutions is that there are implicit assumptions in the algebra about the coefficients being positive. For example in the last problem A needs to be positive, –which you see from solving the ODE $du/dt = 1-u$: it gives $-\log(1-u) = t + C$, and we then convert it to $u = 1 - Ae^{-t}$, with $A = e^{-C}$. On the other hand, if x_0 is larger than π, then our formula implies that $A < 0$–and our solution doesn't work.