APMTH-105 Notes Section #10

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Goals for the week

1. Learn to think qualitatively about how solutions of PDE's combining ODEs with advection.
2. Learn to think qualitatively about how solutions of PDE's combining ODEs with diffusion.
3. Learn to think qualitatively about how solutions of PDE's combining advection with diffusion.
4. Learn how to deal with separation of variable in three dimensions.

Problem 1: Method of Characteristics

Solve

$$xu_x + yu_y = (x+y)u \tag{1}$$

for $y > 1$ under the initial condition that

$$u(x, y=1) = e^{x+1}.$$

Solution:

We start by writing out

$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt} = (x+y)u. \tag{2}$$

We can then see that

$$\frac{dx}{dt} = x \tag{3}$$

and

$$\frac{dy}{dt} = y \tag{4}$$

We solve for x and y yielding $x = Ae^t$ and $y = Be^t$. We let out initial conditions be $y = 1$ when $t = 0$ and $x = s$. Thus, $x = se^t$ and $y = e^t$. We can now solve for

$$\frac{du}{dt} = (x+y)u = (se^t + e^t)u \tag{5}$$

which yields

$$u = Ce^{(s+1)e^t}. \tag{6}$$

From the initial conditions, $u(0,s) = e^{s+1}$, so $C = 1$. Then we transform (s,t) to (x,y), so

$$\boxed{u = e^{x+y}.}$$

Problem 2: Quasilinear 1st order PDE using Charactersistics

Solve

$$uu_x - yu_y = u^2 + u \tag{7}$$

with initial consitions $u(x, y = 1) = e^x$, for $y > 0$.

Solution:

We start by letting $(x, y) \to (s, t)$ which allows

$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt} = u^2 + u. \tag{8}$$

From out given 1st order Order PDE, we find that $\frac{dx}{dt} = u$ and $\frac{dy}{dt} = -y$. We solve for $\frac{dy}{dt}$ first, $y = C_2(s)e^{-t}$. We solve $\frac{du}{dt}$ using Wolfram Alpha, which yields

$$u = \frac{e^{C_1(s)+t}}{1 - e^{C_1(s)+t}}. \tag{9}$$

With this solution for u, we can solve $\frac{dx}{dt}$,

$$x = C_3(s) - \ln(1 - e^{C_1(s)+t}). \tag{10}$$

We then set $x = s$ and $t = 0$ on $y = 1$ for our initial conditions. We get $y(t = 0) = 1 = C_2(s)e^0 \to C_2(s) = 1$, so $y = e^{-t}$. We then find e^x by taking the exponential of both sides of our x equation.

$$e^x = e^{C_3(s) - \ln(1 - e^{C_1(s)+t})} \tag{11}$$

$$e^x = \frac{e^{C_3(s)}}{e^{\ln(1 - e^{C_1(s)+t})}} \tag{12}$$

$$e^x = \frac{e^{C_3(s)}}{1 - e^{C_1(s)+t}}. \tag{13}$$

At $t = 0$,

$$u = e^x = \frac{e^{C_1(s)}}{1 - e^{C_1(s)}}. \tag{14}$$

Equating these two expressions, implies that $C_1(s) = C_3(s)$. We can then note that $u = e^x e^t$, which means that finally

$$u = \frac{e^x}{y}.$$

We can see that the solution breaks down for $y = 0$.