APMTH-105 Notes Section #3

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Goals for the week

1. Learn the method of undetermined coefficients to solve inhomogeneous second order differential equations.
2. Learn about qualitative behavior of systems of first-oder differential equations by finding fixed points, and their stability.
3. Learn about the qualitative behavior of systems of second order equations by characterizing fixed points and their stability, and sketching the dynamics in a phase plane.

Problem 1: Non-homogeneous Equations, Method of Undetermined Coefficients

Find a solution to:

$$y'' + 3y' + 2y = 10e^{3t} \tag{1}$$

First we must find the homogeneous solution by solving for:

$$y'' + 3y' + 2y = 0 \tag{2}$$

Using the ansatz that $y_h = e^{sx}$ we get the characteristic equation:

$$s^2 + 3s + 2 = 0 \tag{3}$$

Where:

$$s = \frac{-3 \pm \sqrt{9-8}}{2} \tag{4}$$

$$s = -1, -2 \tag{5}$$

So, $y_h = Ae^{-1x} + Be^{-2x}$. For $y_p(t)$ we guess $y_p(t) = Ae^{3t}$, because y_p' and y_p'' will retain the same exponential form:

$$y_p'' + 3y_p' + 2y_p = 9Ae^{3t} + 3(3Ae^{3t}) + 2(Ae^{3t}) = 20Ae^{3t} \tag{6}$$

Setting $20Ae^{3t} = 10Ae^{3t}$ and solving for A gives A=1/2; hence,

$$y_p = \frac{e^{3t}}{2} \tag{7}$$

So $y = y_h + y_p$

$$\boxed{y(x) = Ae^{-1x} + Be^{-2x} + \frac{e^{3t}}{2}}$$

Problem 2: Non-homogeneous Equations, Superposition principle of the method of Undetermined Coefficients

Idea 1: Superposition Principle

Let y_1 be a solution to the differential equation

$$ay'' + by' + cy = f_1(t) \tag{8}$$

and y_2 be a solution to

$$ay'' + by' + yc = f_2(t) \tag{9}$$

Then for any constants k_1 and k_2, the function $k_1y_1 + k_2y_2$ is a solution to the differential equation

$$ay'' + by' + cy = k_1f_1(t) + k_2f_2(t) \tag{10}$$

Proof. This is straightforward; by substituting and rearranging we find

$$a(k_1y_1 + k_2y_2)'' + b(k_1y_1 + k_2y_2)' + c(k_1y_1 + k_2y_2) \tag{11}$$

$$= k_1(ay_1'' + b_1' + cy_1) + k_2(ay_2'' + by_2' + cy_2) \tag{12}$$

$$= k_1f_1(t) + k_2f_2(t)e \qquad \blacksquare \tag{13}$$

Find a particular solution to

$$y'' - y = 8te^t + 2e^t \tag{14}$$

A general solution to the associated homogeneous equation is easily seen to be $c_1e^t + c_2e^{-t}$.Thus, a particular solution for matching the nonhomogeneity $8te^t$ has the form $t(A_1t + A_0)e^t$, whereas matching $2e^2$ required the form A_0te^t. Therefore, we can match both with the first form:

$$y_p = t(A_1t + A_0)e^t = (A_1t^2 + A_0t)e^t \tag{15}$$

$$y_p' = (A_1t^2 + A_0)e^t + (2A_1t + A_0)e^t = [A_1t^2 + (2A_1 + A_0)t + A_0]e^t \tag{16}$$

$$y_p'' = [2A_1t + (2A_1 + A_0)]e^t + [A_1t^2 + (2A_1 + A_0)t + A_0]e^t = [A_1t^2 + (4A_1 + A_0)t + (2A_1 + 2A_0)]e^t \tag{17}$$

Thus,

$$y_p'' - y_p = [4A_1t + (2A_1 + 2A_0)]e^t = 8te^t + 2e^t \tag{18}$$

This yields $A_1 = 2, A_0 = -1$,and so

$$\boxed{y_p = (2t^2 - t)e^t}$$

Problem 3: Qualitative Considerations for Variable-Coefficient and Nonlinear Equations

From Nagel's Fundametals of Differential Equations 8th Edition, see attached.

Idea 2: A closer look at free mechanical vibrations

The governing equation of motion is:

$$F_{ext} = my'' + by' + ky \tag{19}$$

Let's focus on the simple case in which $b = 0$ and $F_{ext} = 0$, the so-called undamped, free case. The equation (19) reduces to

$$m\frac{d^2y}{dt^2} + ky = 0 \tag{20}$$

and, when divided by m, becomes

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \tag{21}$$

where $\omega = \sqrt{k/m}$. The auxiliary equation associated with (21) is $r^2 + \omega^2 = 0$, which has complex conjugate roots $\pm\omega i$. Hence, a general solution to (21) is:

$$y(t) = C_1\cos(\omega t) + C_2\sin(\omega t) \tag{22}$$

We can express $y(t)$ in the more convenient form

$$y(t) = A\sin(\omega t + \phi), \tag{23}$$

with $A \geq 0$, by letting $C_1 = A\sin$ and $C_2 = A\cos\phi$. That is,

$$A\sin(\omega t + \phi) = A\cos\omega t\sin\phi + A\sin\omega t\cos\phi \tag{24}$$

$$A\sin(\omega t + \phi) = C_1\cos\omega t + C_2\sin\omega t \tag{25}$$

Solving for A and ϕ in terms of C_1 and C_2, we find

$$A = \sqrt{C_1^2 + C_2^2} \text{ and } \tan\phi = \frac{C_1}{C_2} \tag{26}$$

where the quadrant in which ϕ lies is determined by the signs of C_1 and C_2. This is because $\sin\phi$ has the same sign as $C_1(\sin\phi = C_1/A)$ and $\cos\phi$ has the same sign as $C_2(\cos\phi = C_2/A)$. For example, if $C_1 > 0$ and $C_2 < 0$, then ϕ is in Quadrant II. (Note, in particular, that ϕ is not simply the arctangent of C_1/C_2, which would lie in Quadrant IV.)

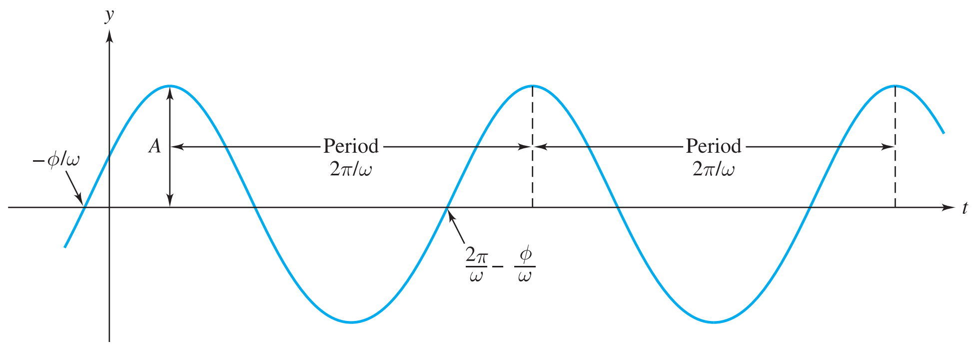

Figure 4.27 Simple harmonic motion of undamped, free vibrations

It is evident from (5) that, as we predicted in Section 4.1, the motion of a mass in an *undamped, free* system is simply a sine wave, or what is called **simple harmonic motion.** (See Figure 4.27.) The constant A is the amplitude of the motion and ϕ is the phase angle. The motion is periodic with **period** $2\pi/\omega$ and **natural frequency** $\omega/2\pi$, where $\omega = \sqrt{k/m}$. The period is measured in units of time, and the natural frequency has the dimensions of periods (or cycles) per unit time. The constant ω is the **angular frequency** for the sine function in (5) and has dimensions of radians per unit time. To summarize:

$$\text{angular frequency} = \omega = \sqrt{k/m} \quad \text{(rad/sec)} ,$$
$$\text{natural frequency} = \omega/2\pi \quad \text{(cycles/sec)} ,$$
$$\text{period} = 2\pi/\omega \quad \text{(sec)} .$$

Observe that the amplitude and phase angle depend on the constants C_1 and C_2, which, in turn, are determined by the initial position and initial velocity of the mass. However, the period and frequency depend only on k and m and not on the initial conditions.

Example 1 A 1/8-kg mass is attached to a spring with stiffness $k = 16$ N/m, as depicted in Figure 4.1. The mass is displaced 1/2 m to the right of the equilibrium point and given an outward velocity (to the right) of $\sqrt{2}$ m/sec. Neglecting any damping or external forces that may be present, determine the equation of motion of the mass along with its amplitude, period, and natural frequency. How long after release does the mass pass through the equilibrium position?

Solution Because we have a case of undamped, free vibration, the equation governing the motion is (3). Thus, we find the angular frequency to be

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{1/8}} = 8\sqrt{2} \text{ rad/sec} .$$

Substituting this value for ω into (4) gives

$$y(t) = C_1 \cos\left(8\sqrt{2}t\right) + C_2 \sin\left(8\sqrt{2}t\right) . \tag{7}$$

Now we use the initial conditions, $y(0) = 1/2$ m and $y'(0) = \sqrt{2}$ m/sec, to solve for C_1 and C_2 in (7). That is,

$$1/2 = y(0) = C_1 ,$$
$$\sqrt{2} = y'(0) = 8\sqrt{2}C_2 ,$$

and so $C_1 = 1/2$ and $C_2 = 1/8$. Hence, the equation of motion of the mass is

$$y(t) = \frac{1}{2}\cos\left(8\sqrt{2}t\right) + \frac{1}{8}\sin\left(8\sqrt{2}t\right) . \tag{8}$$

To express $y(t)$ in the alternative form (5), we set

$$A = \sqrt{C_1^2 + C_2^2} = \sqrt{(1/2)^2 + (1/8)^2} = \frac{\sqrt{17}}{8} ,$$

$$\tan\phi = \frac{C_1}{C_2} = \frac{1/2}{1/8} = 4 .$$

Since both C_1 and C_2 are positive, ϕ is in Quadrant I, so $\phi = \arctan 4 \approx 1.326$. Hence,

$$y(t) = \frac{\sqrt{17}}{8}\sin\left(8\sqrt{2}t + \phi\right) . \tag{9}$$

Thus, the amplitude A is $\sqrt{17}/8$ m, and the phase angle ϕ is approximately 1.326 rad. The period is $P = 2\pi/\omega = 2\pi/\left(8\sqrt{2}\right) = \sqrt{2}\pi/8$ sec, and the natural frequency is $1/P = 8/\left(\sqrt{2}\pi\right)$ cycles per sec.

Finally, to determine when the mass will pass through the equilibrium position, $y = 0$, we must solve the trigonometric equation

$$y(t) = \frac{\sqrt{17}}{8}\sin\left(8\sqrt{2}t + \phi\right) = 0 \tag{10}$$

for t. Equation (10) will be satisfied whenever

$$8\sqrt{2}t + \phi = n\pi \quad \text{or} \quad t = \frac{n\pi - \phi}{8\sqrt{2}} \approx \frac{n\pi - 1.326}{8\sqrt{2}} , \tag{11}$$

n an integer. Putting $n = 1$ in (11) determines the first time t when the mass crosses its equilibrium position:

$$t = \frac{\pi - \phi}{8\sqrt{2}} \approx 0.16 \text{ sec} . \quad ◆$$

In most applications of vibrational analysis, of course, there is some type of frictional or damping force affecting the vibrations. This force may be due to a component in the system, such as a shock absorber in a car, or to the medium that surrounds the system, such as air or some liquid. So we turn to a study of the effects of damping on free vibrations, and equation (2) generalizes to

$$\mathbf{m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0} . \tag{12}$$

The auxiliary equation associated with (12) is

$$mr^2 + br + k = 0 , \tag{13}$$

and its roots are

$$\frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm \frac{1}{2m}\sqrt{b^2 - 4mk} . \tag{14}$$

As we found in Sections 4.2 and 4.3, the form of the solution to (12) depends on the nature of these roots and, in particular, on the discriminant $b^2 - 4mk$.

Underdamped or Oscillatory Motion ($b^2 < 4mk$)

When $b^2 < 4mk$, the discriminant $b^2 - 4mk$ is negative, and there are two complex conjugate roots to the auxiliary equation (13). These roots are $\alpha \pm i\beta$, where

(15) $$\alpha := -\frac{b}{2m}\,, \qquad \beta := \frac{1}{2m}\sqrt{4mk - b^2}\,.$$

Hence, a general solution to (12) is

(16) $$y(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)\,.$$

As we did with simple harmonic motion, we can express $y(t)$ in the alternate form

(17) $$y(t) = Ae^{\alpha t}\sin(\beta t + \phi)\,,$$

where $A = \sqrt{C_1^2 + C_2^2}$ and $\tan \phi = C_1/C_2$. It is now evident that $y(t)$ is the product of an exponential **damping factor,**

$$Ae^{\alpha t} = Ae^{-(b/2m)t}\,,$$

and a sine factor, $\sin(\beta t + \phi)$, which accounts for the oscillatory motion. Because the sine factor varies between -1 and 1 with period $2\pi/\beta$, the solution $y(t)$ varies between $-Ae^{\alpha t}$ and $Ae^{\alpha t}$ with **quasiperiod** $P = 2\pi/\beta = 4m\pi/\sqrt{4mk - b^2}$ and **quasifrequency** $1/P$. Moreover, since b and m are positive, $\alpha = -b/2m$ is negative, and thus the exponential factor tends to zero as $t \to +\infty$. A graph of a typical solution $y(t)$ is given in Figure 4.28. The system is called **underdamped** because there is not enough damping present (b is too small) to prevent the system from oscillating.

It is easily seen that as $b \to 0$ the damping factor approaches the constant A and the quasifrequency approaches the natural frequency of the corresponding undamped harmonic motion. Figure 4.28 demonstrates that the values of t where the graph of $y(t)$ touches the exponential curves $\pm Ae^{\alpha t}$ are close to (but not exactly) the same values of t at which $y(t)$ attains its relative maximum and minimum values (see Problem 13).

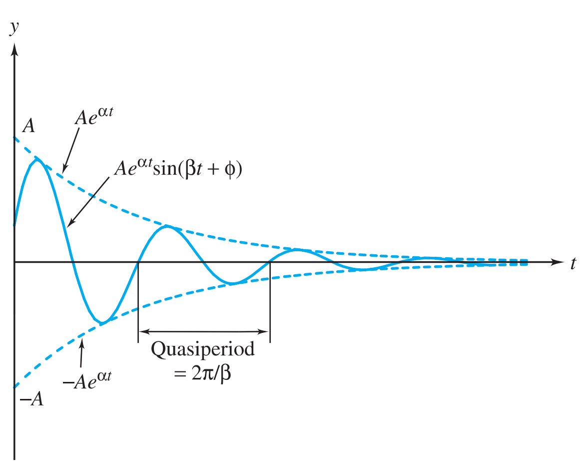

Figure 4.28 Damped oscillatory motion

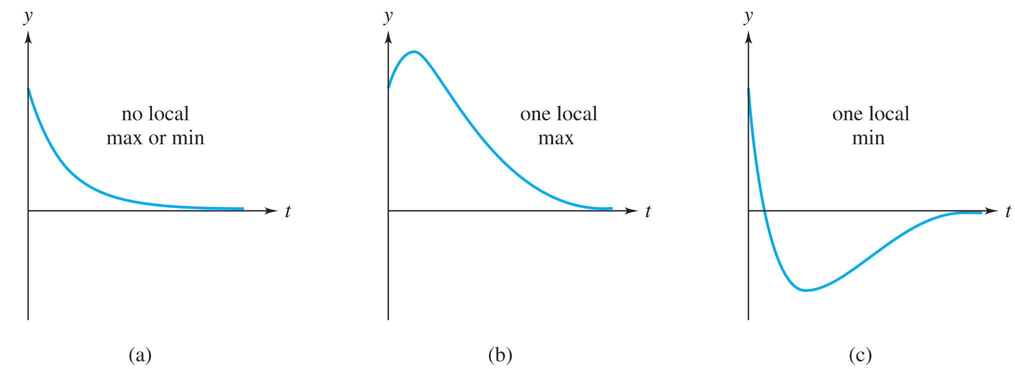

Figure 4.29 Overdamped vibrations

Overdamped Motion ($b^2 > 4mk$)

When $b^2 > 4mk$, the discriminant $b^2 - 4mk$ is positive, and there are two distinct real roots to the auxiliary equation (13):

(18) $$r_1 = -\frac{b}{2m} + \frac{1}{2m}\sqrt{b^2 - 4mk}\ , \qquad r_2 = -\frac{b}{2m} - \frac{1}{2m}\sqrt{b^2 - 4mk}\ .$$

Hence, a general solution to (12) in this case is

(19) $$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}\ .$$

Obviously, r_2 is negative. And since $b^2 > b^2 - 4mk$ (that is, $b > \sqrt{b^2 - 4mk}$), it follows that r_1 is also negative. Therefore, as $t \to +\infty$, both of the exponentials in (19) decay and $y(t) \to 0$. Moreover, since

$$y'(t) = C_1 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t} = e^{r_1 t}\left(C_1 r_1 + C_2 r_2 e^{(r_2 - r_1)t}\right)\ ,$$

we see that the derivative is either identically zero (when $C_1 = C_2 = 0$) or vanishes for at most one value of t (when the factor in parentheses is zero). If the trivial solution $y(t) \equiv 0$ is ignored, it follows that $y(t)$ has at most one local maximum or minimum for $t > 0$. Therefore, $y(t)$ *does not oscillate.* This leaves, qualitatively, only three possibilities for the motion of $y(t)$, depending on the initial conditions. These are illustrated in Figure 4.29. This case where $b^2 > 4mk$ is called **overdamped** motion.

Critically Damped Motion ($b^2 = 4mk$)

When $b^2 = 4mk$, the discriminant $b^2 - 4mk$ is zero, and the auxiliary equation has the repeated root $-b/2m$. Hence, a general solution to (12) is now

(20) $$y(t) = (C_1 + C_2 t)e^{-(b/2m)t}\ .$$

To understand the motion described by $y(t)$ in (20), we first consider the behavior of $y(t)$ as $t \to +\infty$. By L'Hôpital's rule,

(21) $$\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \frac{C_1 + C_2 t}{e^{(b/2m)t}} = \lim_{t\to+\infty} \frac{C_2}{(b/2m)e^{(b/2m)t}} = 0$$

(recall that $b/2m > 0$). Hence, $y(t)$ dies off to zero as $t \to +\infty$. Next, since

$$y'(t) = \left(C_2 - \frac{b}{2m}C_1 - \frac{b}{2m}C_2 t\right)e^{-(b/2m)t} ,$$

we see again that a nontrivial solution can have at most one local maximum or minimum for $t > 0$, so motion is *non*oscillatory. If b were any smaller, oscillation would occur. Thus, the special case where $b^2 = 4mk$ is called **critically damped** motion. Qualitatively, critically damped motions are similar to overdamped motions (see Figure 4.29 again).

Example 2 Assume that the motion of a mass–spring system with damping is governed by

$$\text{(22)} \qquad \frac{d^2y}{dt^2} + b\frac{dy}{dt} + 25y = 0 ; \qquad y(0) = 1 , \qquad y'(0) = 0 .$$

Find the equation of motion and sketch its graph for the three cases where $b = 6$, 10, and 12.

Solution The auxiliary equation for (22) is

$$\text{(23)} \qquad r^2 + br + 25 = 0 ,$$

whose roots are

$$\text{(24)} \qquad r = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 100} .$$

Case 1. When $b = 6$, the roots (24) are $-3 \pm 4i$. This is thus a case of underdamping, and the equation of motion has the form

$$\text{(25)} \qquad y(t) = C_1 e^{-3t}\cos 4t + C_2 e^{-3t}\sin 4t .$$

Setting $y(0) = 1$ and $y'(0) = 0$ gives the system

$$C_1 = 1 , \qquad -3C_1 + 4C_2 = 0 ,$$

whose solution is $C_1 = 1, C_2 = 3/4$. To express $y(t)$ as the product of a damping factor and a sine factor [recall equation (17)], we set

$$A = \sqrt{C_1^2 + C_2^2} = \frac{5}{4} , \qquad \tan\phi = \frac{C_1}{C_2} = \frac{4}{3} ,$$

where ϕ is a Quadrant I angle, since C_1 and C_2 are both positive. Then

$$\text{(26)} \qquad y(t) = \frac{5}{4}e^{-3t}\sin(4t + \phi) ,$$

where $\phi = \arctan(4/3) \approx 0.9273$. The underdamped spring motion is shown in Figure 4.30(a) on page 220.

Case 2. When $b = 10$, there is only one (repeated) root to the auxiliary equation (23), namely, $r = -5$. This is a case of critical damping, and the equation of motion has the form

$$\text{(27)} \qquad y(t) = (C_1 + C_2 t)e^{-5t} .$$

Setting $y(0) = 1$ and $y'(0) = 0$ now gives

$$C_1 = 1 , \qquad C_2 - 5C_1 = 0 ,$$

and so $C_1 = 1, C_2 = 5$. Thus,

$$\text{(28)} \qquad y(t) = (1 + 5t)e^{-5t} .$$

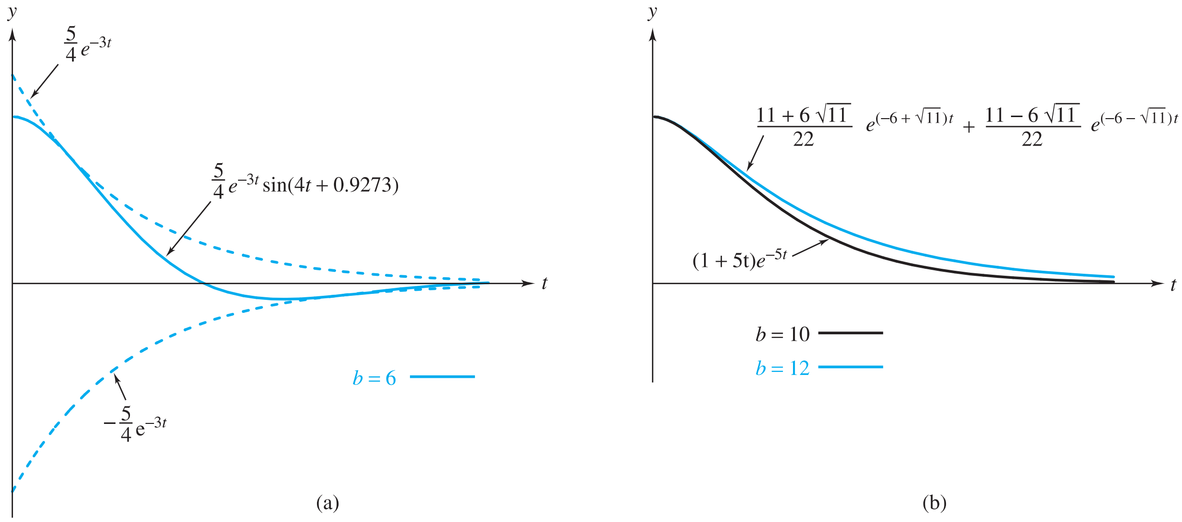

Figure 4.30 Solutions for various values of b

The graph of $y(t)$ given in (28) is represented by the lower curve in Figure 4.30(b). Notice that $y(t)$ is zero only for $t = -1/5$ and hence does not cross the t-axis for $t > 0$.

Case 3. When $b = 12$, the roots to the auxiliary equation are $-6 \pm \sqrt{11}$. This is a case of overdamping, and the equation of motion has the form

(29) $$y(t) = C_1 e^{(-6+\sqrt{11})t} + C_2 e^{(-6-\sqrt{11})t} .$$

Setting $y(0) = 1$ and $y'(0) = 0$ gives

$$C_1 + C_2 = 1 , \qquad (-6 + \sqrt{11})C_1 + (-6 - \sqrt{11})C_2 = 0 ,$$

from which we find $C_1 = (11 + 6\sqrt{11})/22$ and $C_2 = (11 - 6\sqrt{11})/22$. Hence,

(30) $$y(t) = \frac{11 + 6\sqrt{11}}{22} e^{(-6+\sqrt{11})t} + \frac{11 - 6\sqrt{11}}{22} e^{(-6-\sqrt{11})t}$$

$$= \frac{e^{(-6+\sqrt{11})t}}{22} \left\{ 11 + 6\sqrt{11} + (11 - 6\sqrt{11}) e^{-2\sqrt{11}t} \right\} .$$

The graph of this overdamped motion is represented by the upper curve in Figure 4.30(b). ◆

It is interesting to observe in Example 2 that when the system is underdamped $(b = 6)$, the solution goes to zero like e^{-3t}; when the system is critically damped $(b = 10)$, the solution tends to zero roughly like e^{-5t}; and when the system is overdamped $(b = 12)$, the solution goes to zero like $e^{(-6+\sqrt{11})t} \approx e^{-2.68t}$. This means that if the system is underdamped, it not only oscillates but also dies off slower than if it were critically damped. Moreover, if the system is overdamped, it again dies off more slowly than if it were critically damped (in agreement with our physical intuition that the damping forces hinder the return to equilibrium).