APMTH-105 Midterm 1 Review

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Summary

1. First order ODE's and Exact equations
2. Approximate Solutions - Dominant balance
3. Numerical Solutions - Euler's Method
4. Second and Higher order ODE's
5. Linearity and Linear Independence - Wronskian
6. Inhomogeneous equations - finding particular solutions
7. 2D fixed points: phase planes
8. 2D nonlinear phase planes

Reminder

Exact Differential Form

Definition The differential form $M(x, y)dx + N(x, y)dy$ is said to be **exact** in a rectangle R if there is a function $F(x, y)$ such that

(4) $$\frac{\partial F}{\partial x}(x, y) = M(x, y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x, y) = N(x, y)$$

for all (x, y) in R. That is, the total differential of $F(x, y)$ satisfies

$$dF(x, y) = M(x, y)dx + N(x, y)dy .$$

If $M(x, y)dx + N(x, y)dy$ is an exact differential form, then the equation

$$M(x, y)dx + N(x, y)dy = 0$$

is called an **exact equation.**

Test for Exactness

Theorem Suppose the first partial derivatives of $M(x, y)$ and $N(x, y)$ are continuous in a rectangle R. Then

$$M(x, y)dx + N(x, y)dy = 0$$

is an exact equation in R if and only if the compatibility condition

(5) $$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$$

holds for all (x, y) in R.†

Problem 1: Solving an Exact equation

From: Nagel Fundamental of Diff Eq. 8th edition

Solve

$$\frac{dy}{dx} = -\frac{1 + e^x y + xe^x y}{xe^x + 2} \tag{1}$$

which can be rewritten as

$$(1 + e^x y + xe^x y)dx + (xe^x + 2)dy = 0 \tag{2}$$

Solution

Here $M = (1 + e^x y + xe^x y)$ and $N = (xe^x + 2)$. Because

$$\frac{\partial M}{\partial y} = e^x + xe^x = \frac{\partial N}{\partial x} \tag{3}$$

equation (2) is exact. If we now integrate $N(x, y)$ with respect to y, we obtain

$$F(x, y) = \int (xe^x + 2)dy + h(x) = xe^x y + 2y + h(x) \tag{4}$$

When we take the partial derivative with respect to x and substitute for M, we get

$$\frac{\partial F}{\partial x}(x, y) = M(x, y) \tag{5}$$

$$e^x y + xe^x y + h'(x) = 1 + e^x y + xe^x y \tag{6}$$

Thus $h'(x) = 1$, so we take $h(x) = x$. Hence, $F(x, y) = xe^x y + 2y + x$, and the solution to equation (2)is given implicitly by $xe^x y + 2y + x = C$ In this case we can solve explicitly for y to obtain

$$\boxed{y = (C - x)/(2 + xe^x)}$$

Idea 1: Equations of the Form $dy/dx = G(ax + by)$

When the right-hand side of the equation $dy/dx = f(x, y)$ can be expressed as a function of the combination $ax + by$, where a and b are constants, that is,

$$\frac{dy}{dx} = G(ax + by),$$

then the substitution

$$z = ax + by$$

transforms the equation into a separable one. The method is illustrated in the next example.

Problem 2: Solving via substitution

From: Nagel Fundamental of Diff Eq. 8th edition

Solve

$$\frac{dy}{dx} = y - x - 1 + (x - y + 2)^{-1} \tag{7}$$

Solution

The right-hand side can be expressed as a function of x-y, that is,

$$y - x - 1 + (x - y + 2)^{-1} = -(x - y) - 1 + [(x - y) + 2]^{-1} \tag{8}$$

so let $z = x - y$. To solve for dy/dx, we differentiate $z = x - y$ with respect to x to obtain $dz/dx = 1 - dy/dx$, and so $dy/dx = 1 - dz/dx$. Substituting into (7) yields

$$1 - \frac{dz}{dx} = -z - 1 + (z + 2)^{-1} \tag{9}$$

or

$$\frac{dz}{dx} = (z + 2) - (z + 2)^{-1} \tag{10}$$

Solving this separable equation, we obtain

$$\int \frac{z + 2}{(z + 2)^2 - 1} dz = \int dx, \tag{11}$$

with using u, du substitution we get

$$\frac{1}{2} \ln |(z + 2)^2 - 1| = x + C_1 \tag{12}$$

from which it follows that

$$(z + 2)^2 = Ce^{2x} + 1. \tag{13}$$

Finally, replacing z by $x - y$ yields the following as an implicit solution to equation (7)

$$\boxed{(x - y + 2)^2 = Ce^{2x} + 1}$$

Problem 3: Dominant Balance

From: Tziperman 2006 Notes

For each of this problem, determine how the solution $y(x)$ behaves at large x. If the differential equation makes $y(x)$ diverge at finite x then describe the functional form of the divergence.

(a) Consider:

$$\frac{dy}{dx} + y^2 = \frac{3}{x^3 + xcos(x)},$$

with $y(1) = 2$.

(b) Consider:

$$\frac{dy}{dx} - \frac{x^3}{1 + x} y^2 = xe^{-x^2},$$

with $y(2) = 3$.

Solutions

(a) $\frac{dy}{dx} + y^2 = \frac{3}{x^3 + x\cos(x)}$, with $y(1) = 2$.

To understand the behavior for large x we consider all possible dominant balances. For $x \to \infty$ only two terms are of comparable magnitude. The third term will be much smaller.

Trial 1)

$$|\frac{dy}{dx}| = |y^2| \gg |\frac{3}{x^3 + x\cos(x)}|$$

So we consider: $\frac{dy}{dx} = -y^2$

Using separation of variables, we find $\frac{1}{y} = x + C$. Applying initial conditions, our solution is: $y = \frac{1}{x - \frac{1}{2}}$

Now we check if our assumption is valid.

$$|y^2| \sim \frac{1}{x^2}$$

$$|\frac{3}{x^3 + x\cos(x)}| \sim \frac{3}{x^3}$$

For large x:

$$|\frac{1}{x^2}| \gg |\frac{3}{x^3}|$$

Trial 2)

$$|\frac{dy}{dx}| = |\frac{3}{x^3 + x\cos(x)}| \gg |y^2|$$

So we consider: $\frac{dy}{dx} = \frac{3}{x^3 + x\cos(x)} \sim \frac{3}{x^3}$ Using separation of variables, we find $y = -\frac{3}{2x^2} + C$. Applying initial conditions, our solution is: $y = -\frac{3}{2x^2} + \frac{7}{2}$ Now we check if our assumption is valid.

$$y^2 = \frac{9}{4x^4} - \frac{21}{4x^2} + \frac{49}{4}$$

For large x:

$$|\frac{3}{x^3 + x\cos(x)}| \sim \frac{3}{x^3}$$

As $x \to \infty$, $\frac{3}{x^3} \to 0; y^2 \to \frac{49}{4}$.

$$|y^2| \gg |\frac{3}{x^3}|$$

Thus, our assumption in Trial 3 is invalid.

Trial 3)

$$|y^2| = |\frac{3}{x^3 + x\cos(x)}| \gg |\frac{dy}{dx}|$$

So we consider: $y^2 = \frac{3}{x^3 + x\cos(x)} \sim \frac{3}{x^3}$ Now we check if our assumption is valid.

$$|\frac{dy}{dx}| \sim -\frac{3\sqrt{3}}{2x^{5/2}}$$

$$|y^2| \sim \frac{3}{x^3}$$

For large x:

$$|\frac{3\sqrt{3}}{2x^{5/2}}| \gg |\frac{3}{x^3}|$$

Thus, our assumption in Trial 3 is invalid.

Of the three possible balances, only trial 1 is consistent.

(b) $\frac{dy}{dx} - \frac{x^3}{1+x}y^2 = xe^{-x^2}$, with $y(2) = 3$. We use the same approach as in part a.

Trial 1)

$$|\frac{dy}{dx}| = |\frac{x^3}{1+x}y^2| \gg |xe^{-x^2}|$$

So we consider: $\frac{dy}{dx} = \frac{x^3}{1+x}y^2 \sim x^2y^2$ Using separation of variables, we find $-\frac{1}{y} = \frac{1}{3}x^3 + C$. Applying initial conditions, our solution is: $y = \frac{3}{9-x^3}$ Now we check if our assumption is valid.

$$|\frac{x^3}{1+x}y^2| \sim \frac{9}{x^4}$$

For large x:

$$|\frac{9}{x^4}| \gg |xe^{-x^2}|$$

Trial 2)

$$|\frac{dy}{dx}| = |xe^{-x^2}| \gg |\frac{x^3}{1+x}y^2|$$

So we consider: $\frac{dy}{dx} = xe^{-x^2}$ Using separation of variables, we find $y = -\frac{1}{2}e^{-x^2} + C$. Applying initial conditions, our solution is: $y = -\frac{1}{2}e^{-x^2} + 3 + \frac{1}{2}e^{-4}$ Now we check if our assumption is valid.

$$\frac{x^3}{1+x}y^2 = \frac{x^3}{1+x}(-\frac{1}{2}e^{-x^2} + 3 + \frac{1}{2}e^{-4})^2$$

For large x:

$$|\frac{x^3}{1+x}y^2| \sim x^2(-\frac{1}{2}e^{-x^2} + 3 + \frac{1}{2}e^{-4})^2$$

$$|\frac{x^3}{1+x}y^2| \gg |xe^{-x^2}|$$

Thus, our assumption is not valid.

Trial 3)

$$|\frac{x^3}{1+x}y^2| = |xe^{-x^2}| \gg |\frac{dy}{dx}|$$

So we consider: $xe^{-x^2} = -\frac{3}{x^3+x\cos(x)} \sim x^2y^2$. $y^2 = -\frac{1}{x}e^{-x^2}$ Now we check if our assumption is valid.

$$|\frac{dy}{dx}| \sim \sqrt{x}e^{-x^2/2}$$

$$|y^2| \sim \frac{3}{x^3}$$

For large x:

$$|\sqrt{x}e^{-x^2/2}| \gg |xe^{-x^2}|$$

Thus, our assumption in Trial 3 is invalid.

Of all possible balances, only Trial 1 is consistent.

>From Trial 1, $y = \frac{3}{9-x^3}$. Thus, our approximate solution predicts that the function will diverge at $x^3 = 9$ or $x \approx 2.08$

Problem 4: Finding Particular solution of Harmonic oscillator

Consider the equation for a damped driven harmonic oscillator:

$$\frac{d^2y}{dt^2} + \nu\frac{dy}{dt} + \omega_0^2 y = F_0/m\cos(\omega t). \tag{14}$$

Find the particular solution to this equation.

Solution

The solution to the above is discussed extensively in Greenburg, section 3.8. He uses the ansatz for the particular solution

$$y_p = Acos(\omega t) + Bsin(\omega t)$$

that we advocated in our "rules of thumb" in the lecture. Here we will follow a different tact, that uses complex algebra, but which simplifies the algebra considerably. Let's use the ansatz

$$y_p = Ce^{i\omega t}. \tag{15}$$

Now you might object that this particular solution is complex but that we know the final answer will be real–this is true, except that if we remember to take just the real part of this formula at the end then everything will be OK. Indeed, note that $\Re F_0/me^{i\omega t} = F_0/mcos(\omega t)$.
Now, $y_p' = i\omega Ce^{i\omega t}$, and $y_p'' = -C\omega^2 e^{i\omega t}$. So if we plug into our equation we obtain that

$$(-\omega^2 + \nu i\omega + \omega_0^2)C = F_0/m.$$

Hence we have that

$$y_p = \frac{F_0/m}{\omega_0^2 - \omega^2 + \omega\nu i}e^{i\omega t}.$$

We now need to take the real part of this answer. The easiest way to do this is to first write $\omega_0^2 - \omega^2 + \omega\nu i = Re^{i\phi}$, where $R = \sqrt{(\omega_0 - \omega)^2 + \omega^2\nu^2}$ is the magnitude and $tan(\phi) = \omega\nu/(\omega_0^2 - \omega^2)$.
Then we can take the real part and find that

$$y_p = \frac{F_0/m}{\sqrt{(\omega_0 - \omega)^2 + \omega^2\nu^2}}cos(\omega t - \phi).$$