1/26/15 lecture 1

Introduction lecture.

First order diff eq.

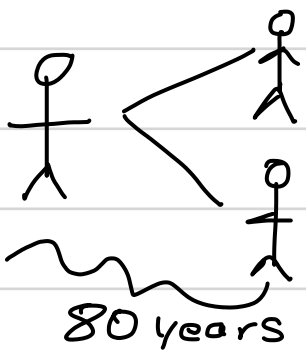
$\frac{dy}{dx} = f(x,y)$ Find $y(x)$ Given: $y(x_0) = y_0$

Notation: $\frac{dy}{dx}, \frac{dy}{dt}, \dot{y}, y'$

Categorization: Linear vs. Nonlinear

$$\frac{dy}{dx} = x^2 y + e^x \qquad y\frac{dy}{dx} = x^3 y^5 + 5$$

The most famous Diff. Eq.



$$\Delta N = \Delta t\, \beta(t) N + \alpha(t)$$

$\beta(t) \to \frac{1}{80 \text{ yrs}}$

$\alpha(t)$ migration

$$\frac{\Delta N}{\Delta t} = \frac{dN}{dt} = \beta N \to \boxed{N(t) = N_0 e^{\beta t}}$$

Suppose $N_0 = 2$

$N(t) = 5 \times 10^9$

$\boxed{t = 1731 \text{ yrs}}$

Doomsday Model

$$\frac{dN}{dt} = \gamma N^2 \to N(t) = \frac{N_0}{1 - t/t^*}$$

1/28/15 lecture 2

Announcements:
- PS0 (optional, MATLAB) due Fri Feb.6
- PS1 up today, due next Wed@ 11am
- Matlab review TBD

$\frac{dN}{dt} = a(t)N \rightarrow \int \frac{dN}{N} = \int a(t)N\,dt$

$$N(t) = N_0 \exp\left[\int_0^t \alpha(t')\,dt'\right] + \underbrace{\frac{1}{t}\int_0^t \alpha(t')\,dt'}_{\text{average}}$$

Solving 1st order ODE

3 Methods

(1) Exact ("analytical") [Today]
(2) Approximate [Fri]
(3) Numerically [Mon.]

(1) Exact Solutions

Bag of tricks

Linear & Homogeneous

$\frac{dy}{dx} + p(x)y = 0$

$\int \frac{dy}{y} = -\int dx\, p(x)$

$\ln(y) = -\int dx' p(x')$

$y_h = e^{-\int dx' p(x')}$

$y_p = A(x) e^{-\int dx' p(x')}$

Plug into (*), $\frac{dy_p}{dx} = \frac{dA}{dx} e^{-\int \cdots} - A p(x) e^{-\int \cdots}$

$\frac{dA}{dx} e^{-\int} - A p e^{-\int} + p A e^{-\int} = q(x)$

$\frac{dA}{dx} e^{-\int p(x')\,dx'} = q(x)$

$\frac{dA}{dx} = q(x) e^{\int p(x')\,dx'}$

$y = y_p + C y_h$

Linear & Inhomogeneous

$\frac{dy}{dx} + p(x)y = q(x)$ *

Example:

$\frac{dy}{dx} + y = \frac{1}{1+x^2}$, $y(x=0) = 2$

Homogeneous: $\frac{dy}{dx} + y = 0 \rightarrow y_h = ce^{-x}$

Particular: $y_p = A(x)\, y_h$

$\frac{dA}{dx} e^{-x} = \frac{1}{1+x^2} \rightarrow \frac{dA}{dx} = \frac{e^x}{1+x^2}$

$A(x) = \int_0^x dx' \frac{e^{x'}}{1+x'^2}$ $\quad y_p = e^{-x} \int_0^x \frac{dx' e^{x'}}{1+x'^2}$

General solution

$y = ce^{-x} + e^{-x} \int_0^x \frac{e^{x'} dx'}{1+x'^2}$, $y(0) = \boxed{2 = c}$

Integrating Factors

$\sigma(x)\left(\frac{dy}{dx} + y\right) = \frac{1}{1+x^2}\sigma(x)$

$\frac{d}{dx}(\sigma(x) y)$ ← would like this to be true

$\frac{d}{dx}(\sigma y) = \sigma \frac{dy}{dx} + \sigma y$

$\frac{d\sigma}{dx} y + \sigma \frac{dy}{dx}$

Need $\frac{d\sigma}{dx} = \sigma$

$\boxed{\sigma(x) = e^x}$

$\int \frac{d}{dx} |e^x y| = \int \frac{e^x}{1+x^2}$

$\int_{y(0)}^{y(x)} d(e^x y) = \int_0^x \frac{e^{x'}}{1+x^2} dx'$

$e^x y(x) - y(0)$

Separation of Variables

$\frac{dy}{dx} = X(x) Y(y) \Rightarrow \int_0^{y(x)} \frac{dy}{Y(y)} = \int_0^x X(x) dx$

From Monday

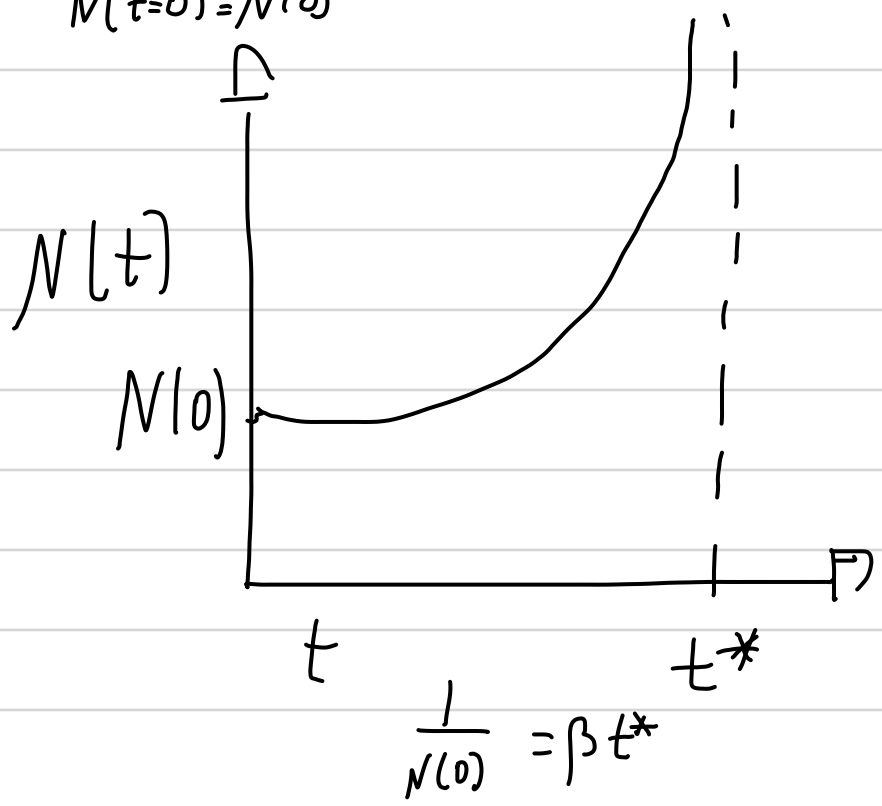
$\frac{dN}{dt} = \beta N^2 \rightarrow \int_{N(0)}^{N(t)} \frac{dN}{N^2} = \int_0^t \beta\, dt'$

$-\frac{1}{N}\Big|_{N(0)}^{N(t)} = \beta t \Rightarrow \frac{-1}{N(t)} + \frac{1}{N(0)} = \beta t$

$\frac{1}{N(0)} - \beta t = \frac{1}{N(t)}$

$\boxed{N(t) = \frac{1}{\frac{1}{N(0)} - \beta t}}$

$N(t=0) = N(0)$



$\frac{1}{N(0)} = \beta t^*$

$\frac{dN}{dt} = \beta N^2 + m$

1/30/15 lecture 3

Announcements:

(1) MATLAB Session –
Sign up on website

(2) Sectioning → Will do on Saturday

(3) Piazza

Last class:

Exact Methods 1st order ODE

Integrating factors

$y' + P(x)y = q(x)$

Step 1: Multiply by $\sigma(x)$

$$\sigma(x)(y' + Py) = \sigma q$$

$$\frac{d}{dx}(\sigma y)$$

Step 2: $\frac{d\sigma}{dx} = p(x)\sigma(x)$

Step 3: $d(\sigma y) = \sigma q(x)$

$$\sigma y = \int \sigma p(x)$$

$$y = \frac{1}{\sigma}\int \sigma p(x)$$

Example: $\frac{dy}{dx} + \underbrace{2x}_{P(x)} y = \underbrace{x}_{q(x)}$

$\frac{d\sigma}{dx} = 2x\sigma \Rightarrow \int \frac{d\sigma}{\sigma} = \int 2x\,dx$

$\log \sigma = x^2 + C \rightarrow \sigma = e^{x^2}$

$\frac{d}{dx}(e^{x^2}y) = xe^{x^2}$

$\int_{y(0)}^{e^{x^2}y} d(e^{x^2}y) = \int_0^x dx' x' e^{x'^2}$

$e^{x^2}y - y(0) = \frac{1}{2}|e^{x^2} - 1|$

$e^{x^2}y - y(0) = \frac{1}{2}(e^{x^2} - 1)$

$y = y(0)e^{-x^2} + \frac{1}{2} - \frac{1}{2}e^{-x^2}$

As $x \to \infty$ $y \to \frac{1}{2}$
For every $y(0)$

Solving $\underset{\text{I}}{\frac{dy}{dx}} + \underset{\text{II}}{2xy} = \underset{\text{III}}{x}$ using Dominant balance

What happens when $X \to \infty$

If term (I) is negligible $\Rightarrow 2xy = x \Rightarrow y = \frac{1}{2}$

Method of Dominant Balance

Suppose you have an equation.

$A+B+C+D=0$

Two are the biggest say A & D $\Rightarrow$ solve $A+D=0$

Examples

(1) $\frac{dy}{dx} + x\tan(y) = x$

(2) $\frac{dy}{dx} + y = \frac{1}{1+x^2}$

$\overset{a}{\frac{dy}{dx}} + x\overset{b}{\tan}(y) = \overset{c}{x}$ $\Rightarrow$ 3 Dominant balance possibilities

$A+B=0$; $|C| \ll |A|, |B|$

$A+C=0$; $|B| \ll |A|, |C|$

$B+C=0$; $|A| \ll |B|, |C|$

Experiment

$A=C$ $\frac{dy}{dx} = x \Rightarrow y = \frac{x^2}{2} + c$

$|B| = x\tan(y), x\tan(x^2)$

$B=C$ $x\tan(y) = x$

$\tan(y) = 1 \Rightarrow y = \frac{\pi}{4}$ | consistent

$A\frac{dy}{dx} = 0$

$\frac{dy}{dx} + y = \frac{1}{1+x^2}$

A B C

$A+B=0$

$\frac{dy}{dx} + y = 0$

$y = Ae^{-x}$

$C = \frac{1}{1+x^2} \approx \frac{1}{x^2}$

$|C| \ll |A, B|$

$B=C$ $y = \frac{1}{1+x^2} \approx \frac{1}{x^2}$

$|A| \ll |B, C|$

$A = \frac{dy}{dx} = -\frac{2}{x^3}$

$|A| = \left|\frac{2}{x^3}\right| \ll \frac{1}{x^2}$ ✓

2/2/15 lecture 4

Bernoulli Equation

$\frac{dy}{dx} + P(x)y = q(x)\,y^n$

$n=1$: $\frac{dy}{dx} = (q(x)-P(x))\,y$

$\int \frac{dy}{y} = \int dx\,(q(x)-P(x)) \rightarrow \ln(y) + c_1 = \int_*^x ds\,(q(s)-P(s))$

$\boxed{y(x) = Ae^{\int_*^x ds(q(s)-P(s))}}$

$n=0$ $\frac{dy}{dx} + P(x)y = q(x)$ $\quad y_h = A\,e^{-\int_*^x P(s)ds}$

$y_p = B(x)\,e^{-\int_*^x P(s)ds}$ $\quad \frac{dy_p}{dx} = \frac{dB}{dx}e^{-S\cdots} - P(x)B(x)\,e^{-S\cdots}$

$\frac{dB}{dx}e^{-S\cdots} - P(x)B(x)e^{-S\cdots} + P(x)B(x)e^{-S\cdots} = q(x)$

$\frac{dB}{dx}e^{-S\cdots} = q(x)$

Any n: $\quad v = y^{-n}$

$\frac{dv}{ds} = \frac{dv}{dy}\frac{dy}{ds} = (1-n)y^{-n}\frac{dy}{ds}$

$\frac{dy}{ds} = \frac{y^n}{(1-n)}\frac{dv}{ds}$

$\frac{y^n}{1-n}\frac{dv}{dx} + (1-n)P(x)y = q(x)\,y^n(1-n)$

$\frac{dv}{dx} + (1-n)P(x)y^{1-n} = q(x)(1-n)$ $\quad v = y^{1-n}$

$\frac{dv}{dx} + (1-n)P(x)v = (1-n)q(x)$

Bernoulli Eq. $n=0$

Ex 1: $\frac{dv}{dx} = 6\frac{y\ln(y)}{x}$

$\int \frac{dy}{y\ln y} = \int \frac{6\,dx}{x}$; $z = \ln y$ $\quad dz = \frac{1}{y}dy$

$\int \frac{dz}{z} = 6\int \frac{dx}{x} \rightarrow \ln z = 6\ln x + c$

$\ln(\ln(y)) = 6\ln x + c$

$\ln(y) = Cx^6$

$\boxed{y = e^{Cx^6}}$

Example 2: $\frac{dy}{dx} + 3y(y+1)\sin(2x) = 0$

$\frac{dy}{3y(y+1)} = dx\sin(2x)$

$\frac{1}{3y(y+1)} = \frac{A}{3y} + \frac{B}{(y+1)} = \frac{A(y+1)+B\,3y}{3(y+1)}$; $A=1$ & $B=-\frac{1}{3}$

$\int \frac{dy}{3y} - \int \frac{dy}{3(y+1)} = -\int dx\sin(2x)$

$\frac{1}{3}(\ln y - \ln(y+1) = \frac{1}{2}\cos(2x)$; $\ln\left(\frac{y}{y+1}\right) = \frac{3}{2}\cos(2x) + C$

$$\frac{y}{y+1} = Ae^{\frac{3}{2}\cos(2x)}$$

$$y(1 - Ae^{\cdots}) = Ae^{\cdots} \quad \rightarrow y = \frac{Ae^{\frac{3}{2}\cos(2x)}}{1 - Ae^{\frac{3}{2}\cos(2x)}}$$

Example 3

$$2xy\frac{dy}{dx} + y^2 = 2x$$

$\frac{dy}{dx} + \frac{y}{2x} = \frac{1}{y}$; $P(x) = \frac{1}{2x}$, $q(x) = 1$, $n = -1$, $v = y^{1-n} = y^2$

$$\frac{dv}{dx} + \frac{1}{x}v = 2$$

Homogeneous Problem

$$\frac{dv}{dx} + \frac{v}{x} = 0$$

$\int \frac{dv}{v} = -\int \frac{dx}{x}$ $\rightarrow \ln(v) = -\ln(x) + C$

$$\boxed{v_h = \frac{A}{x}}$$

Particular solution

$v_p = Bx$ $\quad \frac{dv}{dx} = B$

$B + B = 2 \rightarrow B = 1$, $v_p = x$

$v = v_h + v_p = \frac{A}{x} + x$

$v = y^2$ $\rightarrow \boxed{y = \sqrt{\frac{A}{x} + x}}$

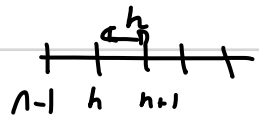
2/4/15 lecture 6

(1): Numerical Methods

(2): Second order ODE's

Chapt 2 Krysig → on iSites

Main Idea: Euler's Method

$\frac{dy}{dx} = f(x,y)$ 

$x_n = hn$

$Y(x_n) = y^n$ $\quad \boxed{y^{n+1} = y^n + h f(x^n, y^n)}$

$\frac{y^{n+1} - y^n}{h} = f(x^n, y^n)$

Questions

(1) How accurate is the method?

(2) How long does it take for code to run?

Accuracy of Euler

$Y(x_{n+1}) = Y(x_n + h)$, Taylor Series

$= Y(x_n) + h\frac{dy}{dx}(x_n) + \frac{h^2}{2!}\frac{d^2y}{dx^2}(x_n$

$y^{n+1} = y^n + h f(x_n, y^n) + \underbrace{\frac{h^2}{2} y'' + \dots}_{O(h^2)}$

Total error = $\underbrace{\text{Error in each step}}_{h^2} \times \#\text{ of steps}$ $\quad \frac{x_{final} - x_{initial}}{h}$

$= O(h)$

4th order Runge-Kutta

$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = h f(x_n, y^n)$

$k_2 = \dots$

$\vdots$

$k_4 = h f(x_{n+1}, y_{n+3})$.

2nd Order ODE's

$y'' + P(x)y' + q(x)y = r(x)$ → Linear

If $r=0$, homogeneous

$y'' = F(x, y, y')$

The most important fact about linear, homogeneous, 2nd order equations

Suppose y_1, y_2 are both solutions

$\Rightarrow y(x) = c_1 y_1 + c_2 y_2$ is also solution

Proof:

$y'' + py' + qy = c_1 y_1'' + c_2 y_2' + (c_1 y_1' + c_2 y_2')p + (c_1 y_1 + c_2 y_2)q$

$= c_1 y_1'' + c_1 p y_1' + c_1 q y_1 + c_2 y_2'' + c_2 p y_2' + c_2 q y_2 = 0$

($c_1 q y_1 = 0$, $c_2 p y_2' = 0$)

Initial Conditions

$$y(x_0) = \boxed{k_0 = C_1 y_1(x_0) + C_2 y_2(x_0)}$$
$$y'(x_0) = \boxed{k_1 = C_1 y_1'(x_0) + C_2 y_2'(x_0)}$$

$y'' + y = 0$

$y_1(x) = \sin(x)$ $\quad y = C_1 \sin(x) + C_2 \cos(x)$

$y_2(x) = \cos(x)$

$y(0) = 1$, $y'(0) = 0$ $\rightarrow y(x) = \cos(x)$

Example.

$y'' = 4y \quad y(1) = 5, \; y'(1) = 0$

Ansatz $y(x) = e^{\alpha x}$

$y'(x) = \alpha e^{\alpha x}$

$y''(x) = \alpha^2 e^{\alpha x}$

$\alpha^2 e^{\alpha x} = 4 e^{\alpha x}$

$\alpha^2 = 4 \rightarrow \alpha = \pm 2$

$y_1(x) = e^{2x} \quad y_2(x) = e^{-2x}$

$\boxed{y = C_1 e^{2x} + C_2 e^{-2x}}$ General Solution

Find C_1 & C_2

$y(1) = \boxed{5 = C_1 e^2 + C_2 e^{-2}}$ $\rightarrow$ Solve

$y'(1) = \boxed{0 = 2C_1 e^2 - 2C_2 e^{-2}}$

2/6/15 Lecture 7

Announcements

PSet 1 Q1a

Should read

"Compute the solution

at $x = \pi$"

Last Class

2nd Order ODE's

$y'' + p(x)y' + q(x)y = r(x)$

When $r=0$ "homogeneous"

$y_1, y_2 \quad y = c_1 y_1 + c_2 y_2$

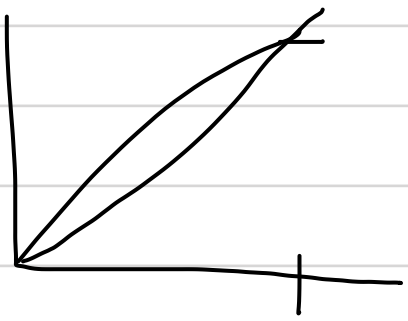
Linear Independence

Definition: Two functions are lin. independent if there does Not exist c_1, c_2 such that $c_1 y_1(x) + c_2 y(x) = 0$

$\sin(x) \cos(x) - e^x, e^{-x} \quad x, x^2$

$c_1 y_1 + c_2 y_2 = 0$

$c_1 y_1' + c_2 y_2' = 0$



$1, 2x|_{x=1}$

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

W = "Wronskian"

$W\vec{c} = \vec{0} \quad \vec{c} = W^{-1}\vec{0} = 0$

$\det W \neq 0$

$W = \begin{pmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{pmatrix}$

$\det W = (e^x)(-e^{-x}) - (e^{-x})e^x$

$= -2$

$$\overline{Y} = a_1 y_1 + a_2 y_2 + a_3 y_3$$

where y_1, y_2, y_3

$$V = C_1 y_1 + C_2 y_2 + C_3 y_3 = 0$$
$$C_1 y_1' + C_2 y_2' + C_3 y_3' = 0$$

$$C_1 y_1'' + C_1 p(x) y_1' + q(x) y_1' + C_2 \cdots$$
$$V'' = 0$$
$$V''' = V'''' = \ldots = 0$$

Solutions with constant coefficients

$$y'' + ay' + by = 0$$

Recall: $y' + y = 0$, $y = e^{-x}$

"Guess"

$y = e^{\lambda x}$ $\quad \lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x} = 0$

$\lambda^2 + a\lambda + b = 0$, $\boxed{\lambda = -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4b}}$

Case I: $a^2 - 4b > 0$

Case II: $a^2 - 4b < 0$

Case III: $a^2 = 4b \rightarrow \lambda = -\frac{a}{2}$

Example 1:

$y'' - 3y' + 2y = 0$, $y(1) = 1$, $y'(1) = 0$

$\lambda_\pm = \frac{3 \pm \sqrt{9-8}}{2}$ $\quad \lambda_+ = 2$, $\lambda_- = 1$; $y = C_1 e^{2x} + C_2 e^x$

$y(1) = 1 = C_1 e^2 + C_2 e^1$

$y'(1) = 0 = 2C_1 e^2 + C_2 e^1 = 0$

$C_1 = \frac{-1}{e^2}$; $C_2 = \frac{2}{e}$

Imaginary #'s Refresher

Eulers Formula

$e^{ix} = \cos x + i \sin x$

Example $y''+16y=0 \quad y(0)=3,\ y'(0)=1$

$y_1=e^{4ix} \qquad y_2=e^{-4ix}$

$$y=C_1e^{i4x}+C_2e^{-i4x}=C_1(\cos 4x+i\sin 4x)+C_2(\cos(4x)+i\sin(4x))$$

$$=\underbrace{(C_1+C_2)}_{\text{"A"}}\cos 4x+i\underbrace{(C_1-C_2)}_{=B}\sin(4x)$$

$$=A\cos(4x)+B\sin(4x)$$

$y''+2y'+y=0 \qquad y(0)=1,\ y'(0)=5$

$\lambda^2+2\lambda+1=0;\ \lambda=-1$

$y=C_1e^{-x}+?$

$y_2=A(x)y_1=A(x)e^{-x}$

$y_2'=A'(x)e^{-x}-A(x)e^{-x}$

$y''=A''(x)e^{-x}-A'(x)e^{-x}-A'(x)e^{-x}+A(x)e^{-x}$

$(A''-2A'+A)e^{-x}+2(A'-A)e^{-x}+Ae^{-x}=0$

$A''=0 \Rightarrow ax+b=A(x)$

$y=C_1e^{-x}+C_2(ax+b)e^{-x}$

$=(d+fx)e^{-x}$

$y'(0)=5 \qquad y'=(d+fx)e^{-x}+fe^{-x}$

$y'(0)=-1+f=5;\ f=6$

2/11/15 lecture 8

$y'' + ay' + by = 0$

$y = e^{\lambda x}$ $\quad \lambda^2 + a\lambda + b = 0$

$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$

I $a^2 > 4b$ 2 real roots

II $a^2 < 4b$ 2 complex roots

III $a^2 = 4b$ $\lambda_1 = \lambda_2$

Case III: double roots

"Reduction of order" P102–103

$y_1 = e^{\lambda x}$, $y_2 = A(x) e^{\lambda x}$

$y_2' = A' e^{\lambda x} + \lambda A e^{\lambda x}$

$y_2'' = A' e^{\lambda x} + 2\lambda A' e^{\lambda x} + \lambda^2 A e^{\lambda x}$

$(A' e^{\lambda x} + 2A'\lambda e^{\lambda x} + \lambda^2 A e^{\lambda x}) + a(A' e^{\lambda x} + \lambda A e^{\lambda x}) + bA e^{\lambda x} = 0$

$A'' e^{\lambda x} + 2A'\lambda e^{\lambda x} + aA' e^{\lambda x} + A(\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + b e^{\lambda x}) = 0$

$A'' + 2A'\lambda + aA' = 0$

$a^2 = 4b \Rightarrow \lambda = -\frac{a}{2}$

$A'' = 0 \Rightarrow y_2 = (c_1 + x) e^{\lambda x}$

$y'' + 4y' + 4y = 0 \quad y(0) = 2, \; y'(0) = 3$

Guess $y_1 = e^{\lambda x}$ $\quad y'(0) = 3$

$\lambda^2 + 4\lambda + 4 = 0 = (\lambda + 2)^2$ $\quad \lambda = -2$, $y_1 = e^{-2x}$ $\quad y_2 = A(x) y_1(x)$

$y_2 = (c_1 + c_2 x) e^{-2x}$

$y = c_1 e^{-2x} + c_2 x e^{-2x}$

$y(0) = c_1 e^0 + (2(0)) e^0 = c_1 = 2$

$y'(0) = -2c_1 e^0 + c_2 e^{-2x0} - 2c_2(0) e^0$

$= -2c_1 + c_2 = 3$

$-4 + c_2 = 3 \Rightarrow c_2 = 7$

$\boxed{y(x) = (2 + 7x) e^{-2x}}$

Inhomogeneous eqn

$y'' + ay' + by = r(x)$

$y = c\,y_h + y_p$

$\uparrow r(x)=0$

Method of undetermined coefficients, G 3.72

Guess form for y_p, that looks like $r(x)$

$r(x)$	Guess y_p
ae^{bx}	Ce^{bx}
$a\sin(kx), b\cos(kx)$	$C_1\sin(kx) + C_2\cos(kx)$
$e^x\sin(x)$, $e^x\cos(x)$	$C_1e^{bx}\cos(kx) + C_2e^{bx}\sin(kx)$
ax^n $(n=0,1,\ldots)$	$C_0 + C_1x + C_2x^2 + \ldots C_nx^n$

Example 1: $r(x) = 3x^2$

$y'' + 4y' + 3y = 3x^2$

homogeneous solution: $y_h = ae^{-x} + be^{-3x}$

particular solution: $C_0 + C_1x + C_2x^2 = y_p$

$y_p' = C_1 + 2C_2x$

$y_p'' = 2C_2$

$2C_2 + 4(C_1 + 2C_2x) + 3(C_0 + C_1x + C_2x^2) = 3x^2$

f	LHS	RHS
x^2	$3C_2$	$= 3$
x^1	$8C_2 + 3C_1$	$= 0$
x^0	$2C_2 + 4C_1 + 3C_0$	$= 0$

$\Rightarrow C_2 = 1$, $C_1 = -\frac{8}{3}$, $C_0 = \frac{26}{9}$

$$y_p = \frac{26}{9} - \frac{8}{3}x + x^2$$

Example 2

$y'' + 4y' + 3y = 4e^x + 2$

$y_p = C_1e^x + C_2$

$y_p' = C_1e^x$, $y_p' = C_1e^x$

$C_1e^x + 4C_1e^x + 3C_1e^x + 3C_2 = 4e^x + 2$

f	LHS	RHS
e^x	$8C_1$	4
e^0	$3C_2$	2

$\Rightarrow C_1 = \frac{1}{2}$; $C_2 = \frac{2}{3}$, Ex 4 $y'' + 2y' = x^2$

$$y_p = (C_0 + C_1x + C_2x^2)x$$

$$y_p' = C_0 + 2C_1x + 3C_2x^2$$

$$y_p'' = 2C_1 + 6C_2x$$

$$2C_1 + 6C_2x + 2(C_0 + 2C_1x + 3C_2x^2) = x^2$$

	LHS	= RHS	
x^2	$6C_2$	$= 1$	$C_2 = \frac{1}{6}$
x^1	$6C_2 + 4C_1$	$= 0$	$C_1 = -\frac{1}{4}$
x^0	$2C_1 + 2C_0$	$= 0$	$C_0 = \frac{1}{4}$

2/13/15 lecture 8

Example: $y'' - y = 3e^x$

$y(0) = 1$

$y'(0) = 0$

$y(x) = y_h(x) + y_p(x)$

$\downarrow$

$c_1 e^x + c_2 e^{-x}$

$y_p(x) = Axe^x \quad y_p'(x) = Ae^x + Axe^x \quad y_p'' = Axe^x + 2Ae^x$

$y_p'' - y_p = Axe^x + 2Ae^x - Axe^x = 3e^x$

$2A = 3 \rightarrow \boxed{A = \frac{3}{2}}$

$y(x) = c_1 e^x + c_2 e^{-x} + \frac{3}{2} x e^x$

$y(0) = 1 = c_1 + c_2$

$y'(0) = 0$

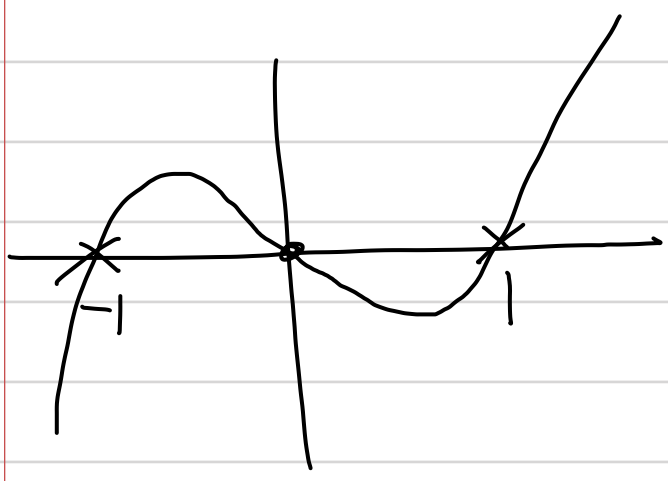
Qualitative behaviour of ODE's (Dynamical systems theory) Steve Strogatz

$\frac{d\vec{y}}{dt} = \vec{f}(y) \quad Y(t)$ Autonomous

$\vec{y} = (y_1(t), y_2(t), y_3(t), \ldots, y_n(t))$

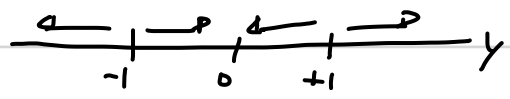
$\frac{dy}{dt} = -y + y^3 = f(y)$

Separable: $\int \frac{dy}{-y+y^3} = \int dt$ Algebra



$0 < y < 1 \quad \frac{dy}{dt} < 0$

$-1 < y < 0 \quad \frac{dy}{dt} > 0$



Fixed points: $f(y^*) = 0$

2 types: $\rightarrow \bullet \leftarrow$ $\leftarrow \bullet \rightarrow$

stable unstable

Dream: $\frac{dy}{dt} = f(y)$

① Find all fixed points $f(y^*) = 0$ [Find all solutions]

② Determine stability

Criterion for Stability

$\frac{dy}{dt} = f(y)$ $\quad y^* + \delta(t)$ $\quad \frac{d}{dt}(y^* + \delta) = \frac{d\delta}{dt} = f(y^* + \delta)$

y^*: fixed point, $f(y^*) = 0$

$= f(y^*) + f'(y^*)\delta + \boxed{\frac{f''(y^*)\delta^2}{2} + \dots}$ small

0

$\frac{d\delta}{dt} = f'(y^*)\,\delta$

$f'(y^*) > 0$ unstable

$f'(y^*) < 0$ stable

$\delta(t) = C e^{f'(y^*)t}$

Higher Order

Rewrite: $m\frac{d^2x}{dt^2} = F$ $\quad \frac{dx}{dt} = v$

2nd order ODE $\quad \frac{dv}{dt} = F/u$

can be written as 2 first order equations

Love affairs (Strogatz)

$R(t)$ = Romeo's love/hate for Juliet

$J(t)$ = Juliet's " " Romeo

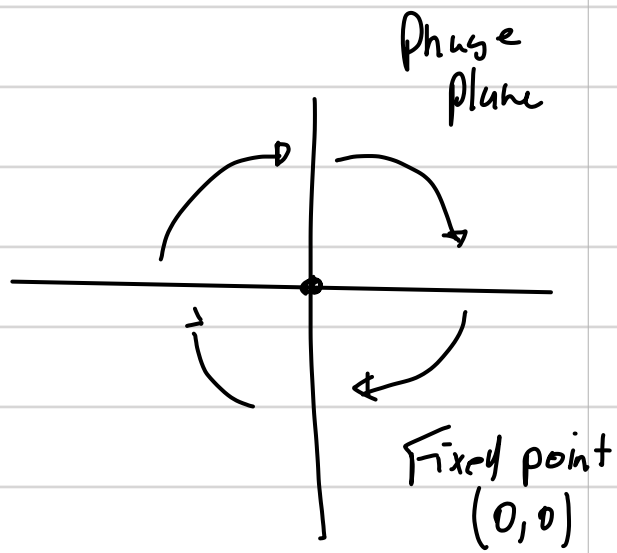
$a, b > 0$

Model #1 $\quad \frac{dR}{dt} = aJ$ $\quad \frac{dJ}{dt} = -bR$

$R(0) = 1$ $\quad J(0) = 1$

$\frac{d^2R}{dt^2} = a\frac{dJ}{dt} = -abR$

$\frac{d^2R}{dt^2} + abR = 0$ $\quad R(t) = A\cos(\sqrt{ab}\,t) + B\sin(\sqrt{ab}\,t)$



Most general model

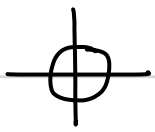
$$\frac{dR}{dt} = aR + bJ$$
$$\frac{dJ}{dt} = cR + dJ$$

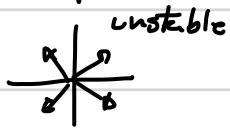
$a > 0 \quad b > 0$
"eager beaver"

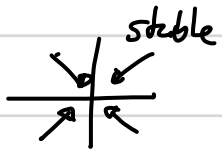
$a < 0 \quad b > 0$
"cautious lover"

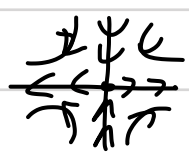
$$\frac{d}{dt}\begin{vmatrix} R \\ J \end{vmatrix} = \begin{vmatrix} 5 & 1 \\ 2 & -3 \end{vmatrix} \begin{vmatrix} R \\ J \end{vmatrix}$$

Possible Fixed points

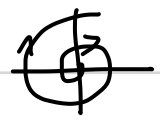







Saddle

unstable spiral


stable spiral

2/18/15 Lecture 9

Qualitative Theory of ODE's

$\frac{dx}{dt} = f(x)$.

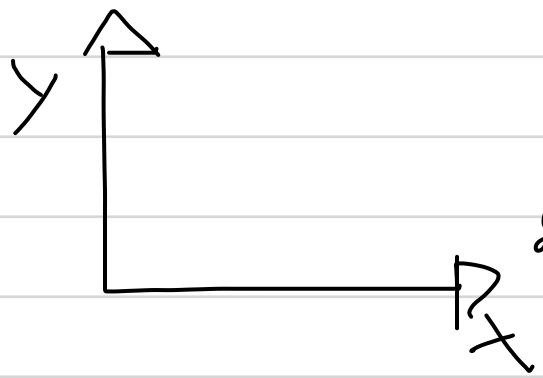
Find pt $f(x^*)=0$ — Phase plane

stability: $f(x,y) = f(x,y)$

$f'(x^*)$

2D $\frac{dx}{dt} = f(x,y)$

$\frac{dy}{dt} = g(x,y)$



Find pts

$f(x^*, y^*) = 0$

$g(x^*, y^*) = 0$

$\frac{dR}{dt} = aR + bJ$

$\frac{dJ}{dt} = cR + dJ$

$\frac{d}{dt}\begin{pmatrix} R \\ J \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}\begin{pmatrix} R \\ J \end{pmatrix}$

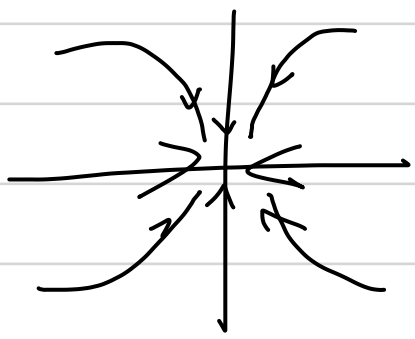
$R(0)$

$J(0)$

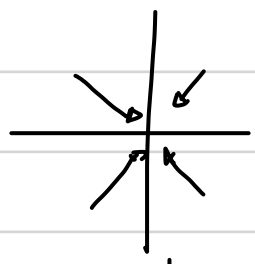
Simple Example

$A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \frac{dR}{dt} = aR \longrightarrow R(t) = R(0)\, e^{at}$

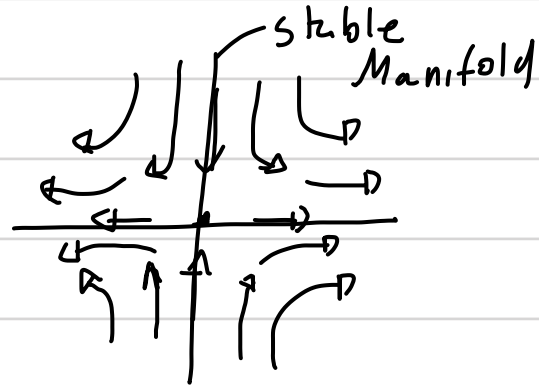
$J(t) = J(0) e^{-t}$



$a < -1$



$a = -1$



$a > 0$

General A.

$\frac{d}{dt}\left(\overrightarrow{\frac{R}{J}}\right) = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{A}\overrightarrow{\begin{pmatrix} R \\ J \end{pmatrix}}$

Dream: Maybe we can find direction where trajectory moves in straight line.

$\frac{d\vec{x}}{dt} = \boxed{A\vec{x} = \lambda\vec{x}}$ Q → Find these directions

$\boxed{(A - \lambda I)}\vec{x} = 0 \qquad \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0 \qquad (a-\lambda)(d-\lambda) - bc = 0$

$$\lambda^2 + \underbrace{(a+d)}_{\tau}\lambda + \underbrace{ad-bc}_{\Delta} = 0$$

$$\lambda_\pm = \frac{\tau \pm (\tau^2 - 4\Delta)^{1/2}}{2}$$

↳ eigenvalues

Eigenvectors

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} R \\ J \end{pmatrix} = \lambda_\pm \begin{pmatrix} R \\ J \end{pmatrix}$$

$$aR + bJ = \lambda_\pm R$$

$$\vec{V_\pm} = \begin{pmatrix} R \\ J \end{pmatrix} = \begin{pmatrix} \frac{\lambda_\pm - a}{b} \\ 1 \end{pmatrix}$$

eigenvector

General Soln

$$\vec{X}(t) = C_1 \vec{V_+} e^{\lambda_+ t} + C_2 \vec{V_-} e^{\lambda_- t}$$

$$\vec{X}(0) = C_1\vec{V_+} + C_2\vec{V_-}$$

$$\begin{pmatrix} R(0) \\ J(0) \end{pmatrix}$$

Example: Strogatz Ex 5.2.1

$\dot{x} = x - y \quad x_0 = 2$

$\dot{y} = 4x - 2y \quad y_0 = -3$

$$A = \begin{bmatrix} 1 & -1 \\ 4 & -2 \end{bmatrix} \qquad \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix}$$

$$\lambda^2 + \lambda - 6 = 0$$

$$\lambda_+ = 2, \ \lambda_- = -3$$

Eigenvectors:

$$\begin{vmatrix} 1 & 1 \\ 4 & -2 \end{vmatrix}\begin{vmatrix} x \\ y \end{vmatrix} = 2\begin{vmatrix} x \\ y \end{vmatrix}$$

$x + y = 2x \rightarrow y = x \quad \vec{V_+} = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$$

$x + y = -3 \quad y = -4x \quad \vec{V_-} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

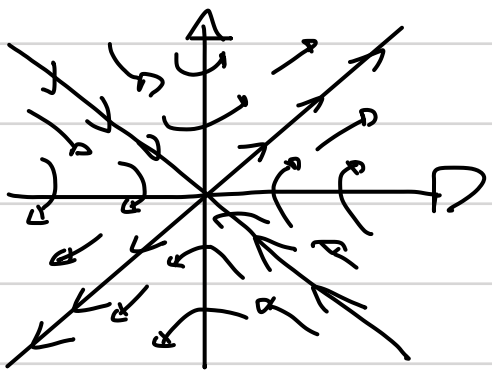
$$\vec{X}(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + C_2\begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$$

$$X(t) = C_1 e^{2t} + C_2 e^{-3t}$$

$$\left.\begin{aligned} X(0) &= C_1 + C_2 = 2 \\ Y(0) &= C_1 - 4C_2 = -3 \end{aligned}\right\} \begin{aligned} C_1 &= 1 \\ C_2 &= 1 \end{aligned}$$

$$\vec{X}(t) = 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + 1\begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$$

Phase Portrait



SADDLE

2/20/15 Lecture 10

Last class

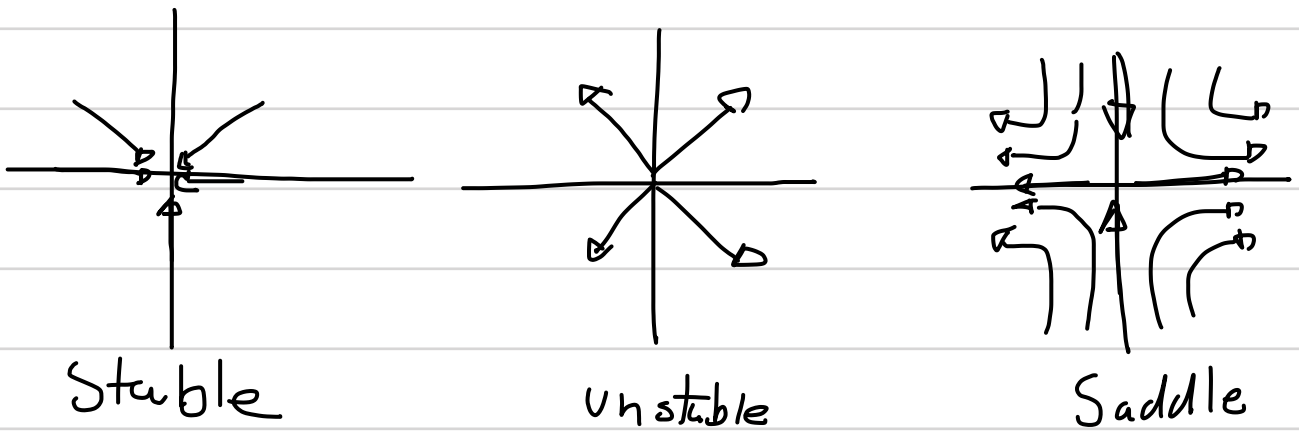
$\frac{d\vec{x}}{dt} = A\vec{x}$; $\vec{x}(0)$ $A\vec{x} = \lambda\vec{x}$ λ_+ $\vec{V}_+$

$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $\vec{x} = \begin{vmatrix} x(t) \\ y(t) \end{vmatrix}$

$\lambda^2 \underbrace{(a+d)}_{\tau}\lambda + \underbrace{ad-bc}_{\Delta} = 0$ $\lambda_\pm = \frac{\tau \pm (\tau^2 + \Delta)^2}{2}$

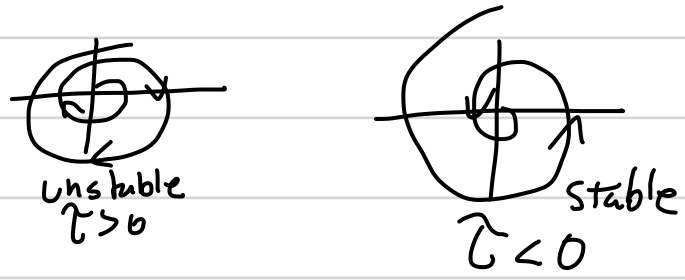
Stability Dictionary

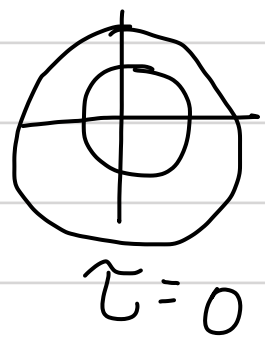
$\lambda_\pm$ – real $\tau^2 > 4\Delta$



$\lambda_\pm$ imaginary $\tau^2 < 4\Delta$

$\lambda_\pm = \frac{\tau}{2} \pm i\left(\frac{\sqrt{4\Delta - \tau^2}}{2}\right)$





Non-linear Problems

$\dot{x} = f(x,y)$

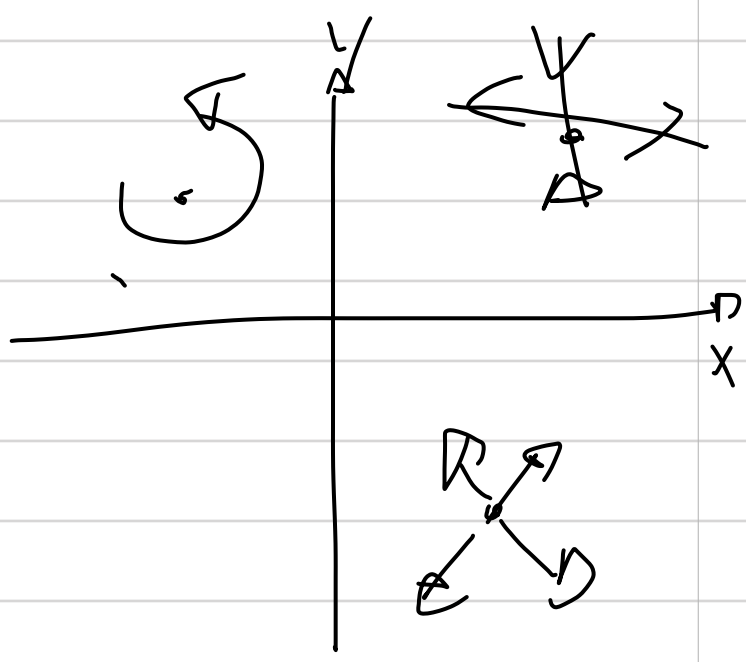
$\dot{y} = g(x,y)$

Agorithm

(1) Find fixed pts $f(x^*, y^*) = 0$

(2) Stability? $g(x^*, y^*) = 0$

(3) Put together



Nullcline

$\frac{dx}{dt}=0 \quad f(x,y)=0$

$\frac{dy}{dt}=0 \quad g(x,y)=0$

Ex: Lotka Volterra Model

$x(t) \to$ #of rabbits

$y(t) \to$ #of foxes

$$\frac{dx}{dt}=\alpha x-\beta xy = f$$

$$\frac{dy}{dt}=\delta xy-\gamma y = g$$

Fixed pts

$f=g=0$

$\alpha=2 \ , \beta=1$
$\gamma=1/2 \ , \delta=1/2$

$\alpha x-\beta xy=0 \qquad \alpha=\beta y \to y=\frac{\alpha}{\beta}, \ x=0$

$\gamma xy-\gamma y=0 \qquad y=0 \ \ \delta x-\alpha=0 \Rightarrow \boxed{x=\frac{\gamma}{\delta}}$

$(x^*,y^*)=(0,0)$

$$(x^*,y^*)=\left(\frac{\gamma}{\delta},\frac{\alpha}{\beta}\right)=(1,2)$$

$\frac{dx}{dt}=f(x,y) \qquad x(t)=x^*+\delta_x(t)$

$\frac{dy}{dt}=g(x,y) \qquad y(t)=y^*+\delta_y(t)$

$$\frac{d\delta_x}{dt}=f(x^*+\delta_x,y^*+\delta_y)=f(x^*,y^*)+\delta_x\,\partial_x f(x^*,y^*)+\delta_y\,\partial_y f(x^*,y^*)$$

$$\frac{d\delta_y}{dt}=g(x^*+\delta_x,y^*+\delta_y)=g(x^*,y^*)+$$

$$\frac{d}{dt}\begin{pmatrix}\delta_x\\ \delta_y\end{pmatrix}=\underbrace{\begin{pmatrix}\partial_x f & \partial_y f\\ \partial_x g & \partial_y g\end{pmatrix}}_{\text{Jacobian}}\begin{pmatrix}\delta_x\\ \delta_y\end{pmatrix}$$

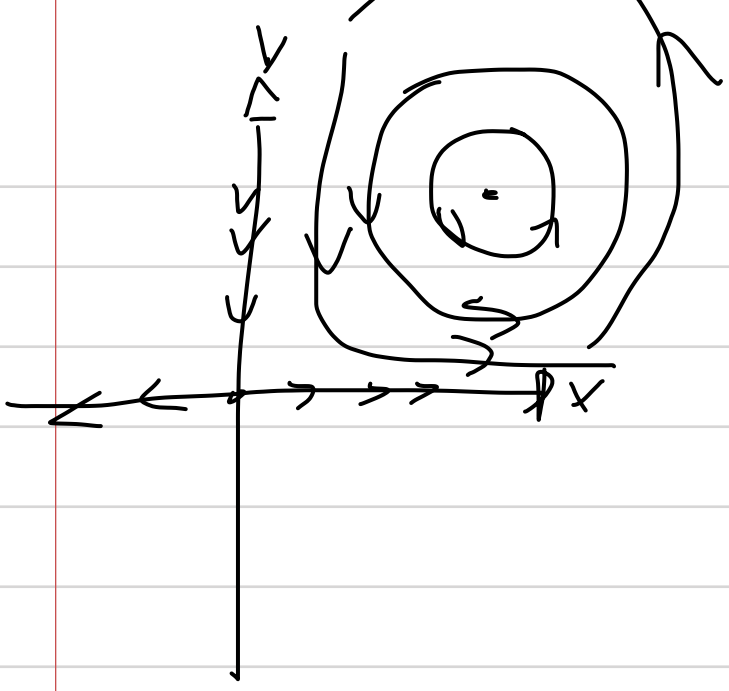
$$J=\begin{vmatrix}\alpha-\beta y & -\beta x\\ \delta y & \delta x-\gamma\end{vmatrix}$$

$(x^*,y^*)=(0,0) \qquad J=\begin{pmatrix}\alpha & 0\\ 0 & -\gamma\end{pmatrix} \Rightarrow \lambda=2,-\frac{1}{2} \qquad$ Saddle

$(1,0) \ (0,1)$

$(x^*,y^*)=(1,2) \qquad J=\begin{vmatrix}0 & -1\\ 1 & 0\end{vmatrix} \Rightarrow \det(J-\lambda I)=(\lambda^2+1)=0$

$\boxed{\lambda=\pm i}$


y
x

2/23/15 lecture 11

Example: $\frac{dx}{dt} = y + x - x^3$, $\frac{dy}{dt} = -y$ || (1) Find FPs (2) Characterize Stability (3) Draw phase planes [Nullclines]

(1) Find Fixed points

$\dot{y} + x - x^3 = 0 \Rightarrow x - x^3 = 0$

$\dot{y} = 0$ $\quad (1 - x^2)x = 0 ; x = 0, +1, -1$

$(x^*, y^*) = (0,0), (1,0), (-1,0)$

(2) Stability $J = \begin{pmatrix} \partial_x \dot{x} & \partial_y \dot{x} \\ \partial_x \dot{y} & \partial_y \dot{y} \end{pmatrix} = \begin{pmatrix} 1 - 3x^2 & 1 \\ 0 & -1 \end{pmatrix}$

$(0,0) \Rightarrow J = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$

$\det(J - \lambda I) = 0$

$\det\begin{pmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{pmatrix} = 0 \quad -(1-\lambda)(1+\lambda) = 0$

$\lambda = 1, -1$

Saddle

$(1,0) \quad (-1,0)$

$(1,0) \Rightarrow J = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$

$\lambda = -2, -1$

Stable

Eigenvectors

$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

$x + y = \lambda x$

$\lambda = +1 \quad x + y = x \rightarrow y = 0 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = -1 \quad x + y = -x \quad y = -2x \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda = -2 \quad -2x + y = -2x \Rightarrow y = 0 \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = -1$

$-2x + y = -x \Rightarrow x = y \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Null clines

$\frac{dx}{dt}=0 \Rightarrow y+x-x^3=0$

$y=x^3-x$

$\frac{dy}{dt}=-y=0$

$\frac{d\vec{x}}{dt}=\vec{f}(\vec{x})$ n equations

$\frac{dx}{dt}=x-y-x(x^2+y^2)$

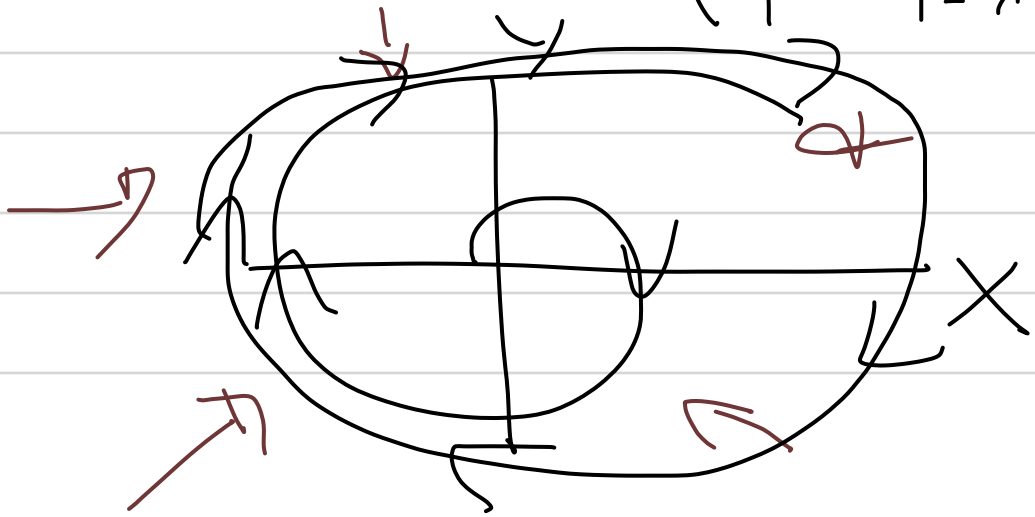
$\frac{dy}{dt}=x+y-y(x^2+y^2)$ $(0,0)$

$$J=\begin{bmatrix} 1-3x^2-y^2 & -1-2yx \\ 1-2xy & 1-3y^2 \end{bmatrix}$$

$$J=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \qquad -x^2$$

$$\det(J-\lambda I)=\det\begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix}=(1-\lambda)^2+1=0$$

$1-\lambda=\pm i \Rightarrow \lambda=1\pm i$



Limit Cycle

$$r^2 = x^2 + y^2$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = \frac{1}{2}\frac{d}{dt}(x^2+y^2) = x\left((r-y-x(x^2+y^2)\right) + y(x+y) - y(x^2+y^2)\Big]^{r^2}$$

$$= x^2 - xy - x^2r^2 + xy + y^2 - y^2r^2$$
$$= x^2 + y^2 - r^2(x^2+y^2)$$
$$= r^2 - r^4$$

$$\frac{1}{2}\frac{d}{dt}r^2 = r^2 - r^4$$
$$r\frac{dr}{dt} \qquad \frac{dr}{dt} = r - r^3$$

Van der Pol

$$\frac{d^2x}{dt^2} + (x^2-1)\frac{dx}{dt} + x = 0$$

2/25/15 lecture 12

Section 2

(i) Main idea of section 2

(ii) A terrible calculation

---

2 themes:

→ Boundary value problem

$y'' = f(x, y')$ $\quad y(0)$ $\quad y(5)$

$\nabla^2 u = 0$

$(\partial_{xx} + \partial_{yy}) u = 0$

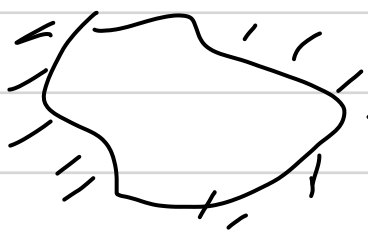
Eigenvalue problem

$y'' + \lambda y = 0$

$\sin(x), \cos(x)$

$y(0) = 0$

$y(-1) = 0$



$y'' + p(x) y' + q(x) y = 0$

Special functions

- Bessel functions
- Legendre functions
- Lagrange functions
- Hermite polynomials

Sturm-Liouville Theory

Example: $y'' + \frac{1}{x-1} y' + 2y = 0 \qquad y(0) = 5, \; y'(0) = 0$

Idea: Write $y(x) = \sum_{n=0}^{\infty} a_n x^n$, pray: that series converges

Background:

$S_N = \sum_{n=0}^{N} a_n x^n$

→ Series converges if $S_N \to$ limit

→ Comparisson test $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

$$\left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| < 1 \qquad |x| < \left| \frac{a_n}{a_{n+1}} \right|$$

radius of convergence

$e^x = 1 + x + \frac{x^2}{2!} + \dots$ $\qquad a_n = \frac{1}{n!}$

$$\frac{a_n}{a_{n+1}} = \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} = n+1$$

Theorem: 4.2.4 Greenberg

Radius of convergence of $y(x)$ is determined by the radius of convergence of $p(x)$ & $q(x)$

$$(x-1)y'' + y' + 2(x-1)y = 0$$

(1) $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\boxed{(x-1)\left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right)}$$
$$\boxed{+\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)}$$
$$\boxed{+2(x-1)\left(\sum_{n=0}^{\infty} a_n x^n\right)} = 0$$

① Rewrite it as

$(\quad)x^2 + (\quad)x$

$+(\quad)x^3 + (\quad)$

$-2(1)a_2 + a_1 - 2a_0$

$-2a_2 + a_1 - 2a_0$

$2a_2 = 2a_0 - a_1$

$\boxed{a_2 = 5}$

$y(0) = a_0 = 5$

$y'(0) = a_1 = 0$

2/27/15 lecture 12

$y'' + p(x)y' + q(x)y = 0$

Last lecture

Power series solutions.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{Radius of convergence}$$

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \qquad (x-1)y'' + y' + 2(x-1)y = 0 \;;\; y(0) = 5 \;\; y'(0) = 0$$

$$y'' - \frac{1}{x-1}y' + 2y = 0$$

$$-(1 + x + x^2 + \dots)$$

$$(x-1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} + 2(x-1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$m = n-1$

$$\sum_{\substack{n=2 \\ m=n-1 \\ n=m+1}}^{\infty} n(n-1)a_n x^{n-1} - \sum_{\substack{n=2 \\ m=n-2}}^{\infty} n(n-1)a_n x^{n-2} + \sum_{\substack{n=0 \\ m=n+1}}^{\infty} 2a_n x^{n+1} - \sum_{\substack{n=1 \\ m=n}}^{\infty} 2a_n x^n = 0$$

$$\sum_m (m+1)m a_{m+1} x^m - \sum_m (m+2)(m+1)a_{m+2} x^m + \sum_m (m+1)a_{m+1} x^m + \sum_m 2a_{m-1} x^m - \sum 2a_m x^m = 0$$

$$(m+1)m a_{m+1} - (m+2)(m+1)a_{m+2} + (m+1)a_{m+1} + 2a_{m-1} - 2a_m = 0$$

$$a_{m+2} = \frac{(m+1)m a_{m+1} + (m+1)a_{m+1} + 2a_{m-1} - 2a_m}{(m+2)(m+1)} \quad \text{for large } m$$

Test for convergence.

Eigenvalue Problems

$y'' + y = 0 \Rightarrow y = A\sin(x) + B\sin(x)$

$y(0) = 0 \qquad y(0) = 0 \Rightarrow \boxed{B = 0}$

$y'(\pi/2) = 0 \qquad y' = A\cos(x) = 0$

$y'' + \lambda^2 y = 0 \quad y(0) = 0 \quad$ Let $y(x) = A\sin(\lambda x) + B\cos(\lambda x)$

$y(l) = 0$

$y(0) = B = 0$

$y(l) = \boxed{A\sin\lambda l = 0}$ non-zero solution $\Rightarrow \sin\lambda l = 0$

$\lambda l = n\pi$

$\lambda_n = \frac{n\pi}{\ell}$ ← eigenvalues

$\frac{d^2y}{dx^2} = -\lambda^2 y$

$\left[\frac{d^2}{dx^2}\right] y = -\lambda^2 y$

matrix $\infty \times \infty$

$y'' + \lambda^2 y = 0$

$y(0) = 0$

$y'(1) = -\alpha y(1)$ $\quad \alpha > 0$

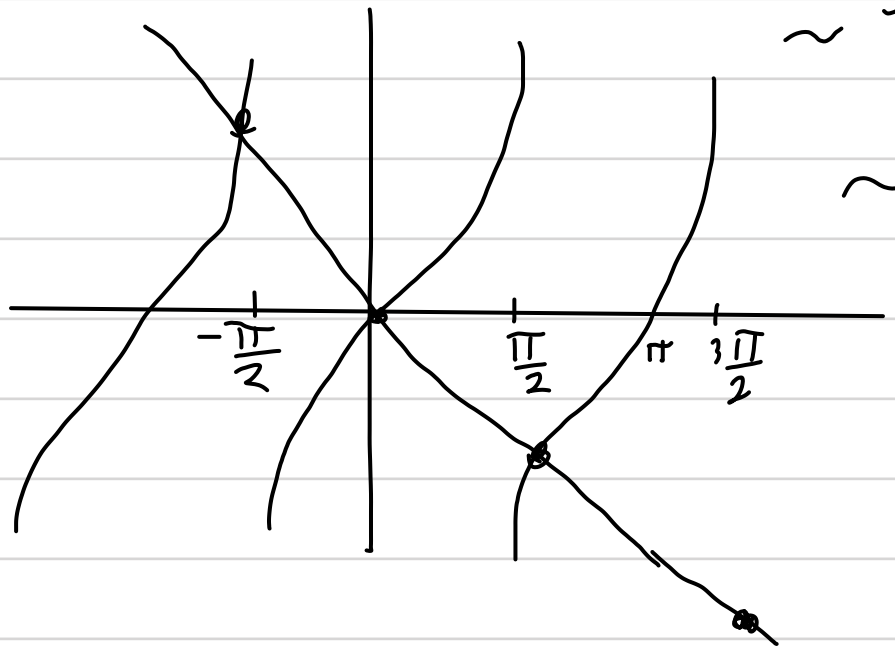
$y(x) = A \sin(\lambda x)$

$y'(1) = A\lambda \cos(\lambda) = -\alpha A \sin(\lambda)$

$A[\lambda \cos(\lambda) + \alpha \sin(\lambda)] = 0$

$\lambda \cos(\lambda) = -\alpha \sin(\lambda)$

$\tan(\lambda) = -\frac{\lambda}{\alpha}$



$\sim \frac{\pi}{2}$

$\sim \frac{3\pi}{2}$

$\sim \frac{5\pi}{2}$

3/4/15 lecture 13

Last class

Eigenvalue problems

$y'' + \lambda^2 y = 0 \quad y(0) = 0; \; y(L) = 0$

$y = A\sin(\lambda y) + B\cos(\lambda y)$

$y(L) = A\underbrace{\sin \lambda L}_{=0} \quad \lambda L = n\pi \quad \lambda_n = \frac{n\pi}{L}$

$y_n = A\sin\frac{n\pi x}{L}$

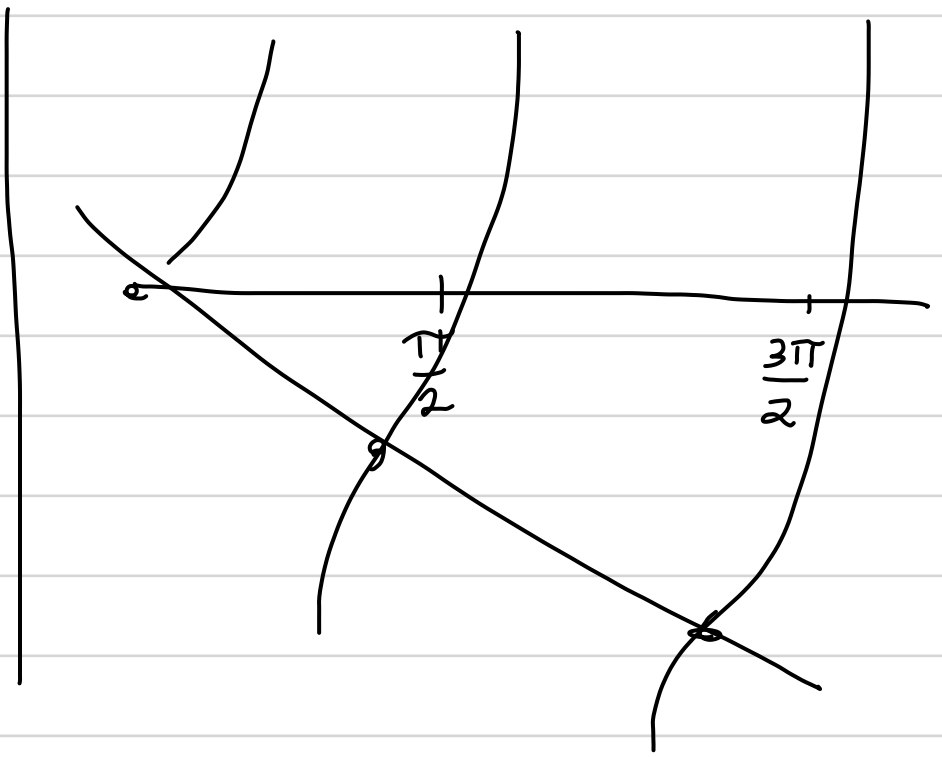
$y'' + \lambda^2 y = 0 \quad y(0) = 0$

$\alpha y(1) + y'(1) = 0$

$y = A\sin\lambda x$

$\alpha A\sin\lambda + A\lambda\cos\lambda = 0$

$\frac{\sin\lambda}{\cos\lambda} = \frac{-\lambda}{\alpha}$



$(n+\frac{1}{2})\pi \approx \lambda_n$

$\lambda_n = \underbrace{(n+\frac{1}{2})\pi}_{a_n = (n+\frac{1}{2})\pi} + \delta$

$\tan\lambda_n = \frac{\sin(a_n+\delta)}{\cos(a_n+\delta)} = \frac{\sin a_n\cos\delta + \cos a_n\sin\delta}{\cos(a_n)\cos\delta - \sin a_n\sin\delta}$

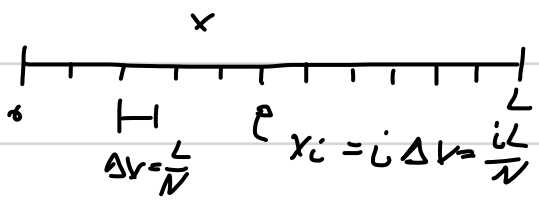
$\tan\lambda_n \approx -\frac{\cos\delta}{\sin\delta} \simeq -\frac{1}{\delta}$

$\tan\lambda_n = -\frac{a_n}{\alpha} \approx \frac{-(n+\frac{1}{2})\pi}{\alpha} = -\frac{1}{\delta} \qquad \delta = \frac{\alpha}{(n+\frac{1}{2})\pi}$

$\lambda_n = (n+\frac{1}{2})\pi + \frac{\alpha}{(n+\frac{1}{2})\pi}$

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0$$

$$M\vec{y} = \lambda\vec{y}$$



$$y_i = y(x_i)$$

$$\left.\frac{d^2y}{dx^2}\right|_{x_i} = \frac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2}$$

$$y_{i\pm1} = y(x_i \pm \Delta x) = y_i \pm y_i' \Delta x + y_i'' \frac{\Delta x^2}{2} + \dots$$

$$y_{i+1} + y_{i-1} = 2y_i + y_i'' (\Delta x)^2$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}$$

$$\frac{1}{(\Delta x)^2}\begin{bmatrix} 0 & 1 & -2 & 1 & 0 \end{bmatrix}$$

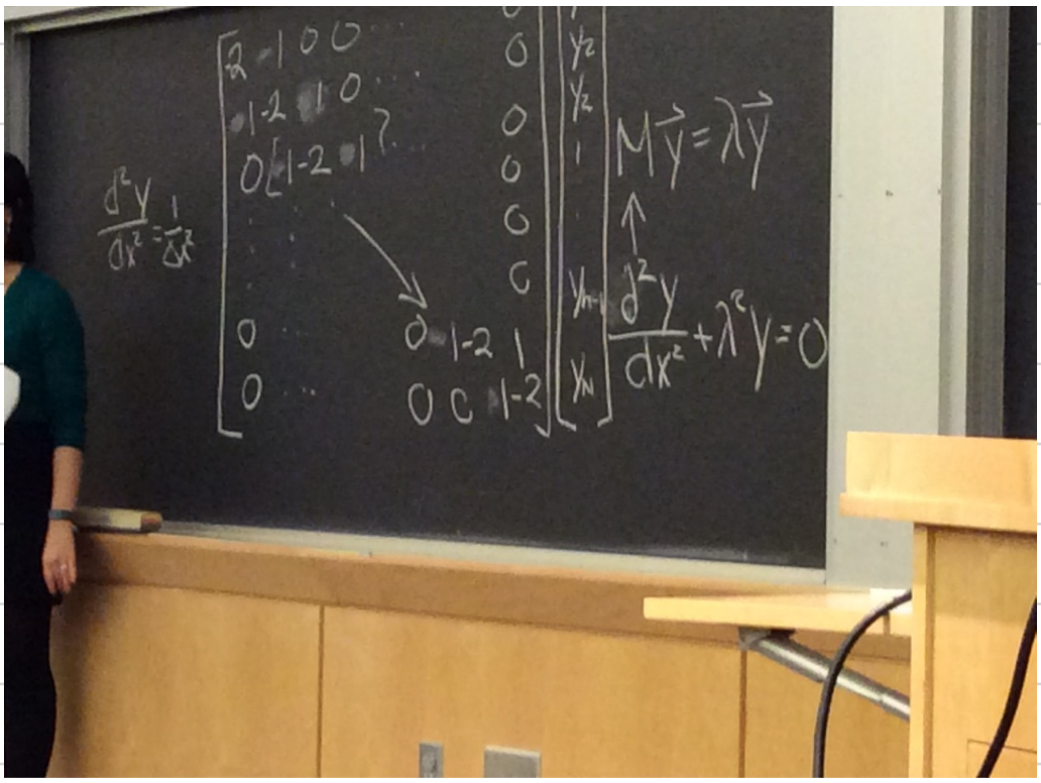
$y(0) = 0$

$$y_1'' = y_2 - y_1 + \cancel{y_0} \to 0$$

$$y_N'' = \frac{y_{N+1} - y_N + y_{N-1}}{\Delta x^2}$$

$y(L) = 0$

$y_{N+1} = 0$



$$\alpha\, y(1) + y'(1) = 0$$

$$\alpha\, y_n + \frac{y_n - y_{n-1}}{\Delta x} = 0$$

$$y_n = \frac{y_{n-1}}{1 + \alpha \Delta x}$$

3/6/15 lecture 14

Did not attend class

3/9/15 lecture 15

Slides

$\vec{v} = \begin{vmatrix} \vdots \\ \vdots \end{vmatrix}$ ← n rows $\vec{v} = \sum_{i=1}^{n} c_i \hat{e}_i$ $i^{th} \to \begin{pmatrix} v \\ \vdots \end{pmatrix}$

$\hat{e}$ unit vector in $\mathbb{R}^n$

$c_i = \vec{v} \hat{e}_i$

$f = \sum_n a_n \phi_n(x)$

$\langle f(x), \phi_m(x) \rangle = \sum_n a_n \langle \phi_n, \phi_m \rangle = a_n \langle \phi_m, \phi_n \rangle$

How fast do Fourier Series converge?

The smoother the function, the faster the series converges

Related to continuity of function and its derivatives

Gibbs phenomenon, explains why oscillation at discontinuities

$$a_n = \frac{\langle f, \phi_n \rangle}{\langle a_n, \phi_n} = \frac{\int_{-1}^{1} f(x) \phi_n(x)\, dx}{\int_{-1}^{1} \phi_n^2(x)\, dx}$$

If there is a discontinuity in the function, convergence goes as $\frac{1}{n}$

If there is a " " " derivative of the function, convergence goes as $\frac{1}{n^2}$ ...

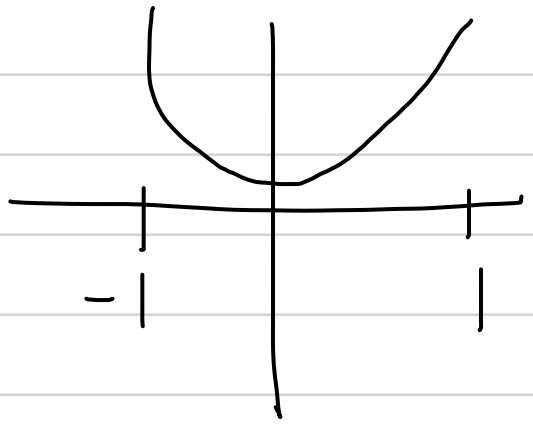
$$a_n = \frac{2}{L} \int_{-L}^{L} f(x) \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{d\left(\frac{-\cos(n\pi x/L)}{n\pi L}\right)}$$

3/11/15 lecture 16

Brief review

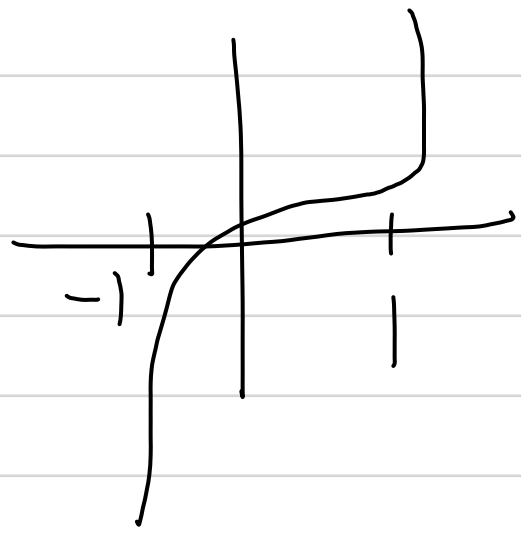
$y'' = -\lambda^2 y$ + B.C.

(ϕ_n, λ_n) $\langle \phi_n, \phi_n \rangle = 0$ $n \neq m$



$f(x) = \sum a_n \phi_n$ $a_n = \frac{\langle F, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}$

$f(x) = x^2 = \sum a_n \cos n\pi x$ $a_n = \frac{4(-1)^n}{\pi^2 n^2}$



$f(x) = \sum b_n \sin n\pi x$

$b_n = \frac{(-1)^n}{n\pi} + \frac{(\ )}{(n\pi)^2}$

Monday:

(1) Take $f(x)$ & periodically extend it to the entire real line. Call this $\tilde{f}$

(2) If $\tilde{f}$ has discontinuities, $a_n \sim \frac{1}{n}$

$\tilde{f}'$ " " , $a_n \sim \frac{1}{n^2}$

$\tilde{f}''$ " " , $a_n \sim \frac{1}{n^3}$

Gibbs phenomenon: $f(x) = \begin{cases} 0 & -1 < x < 0 \\ 1-x & 0 < x < 1 \end{cases}$

Sturm Louiville Theory

$y'' + \lambda^2 y = 0$

Sin & cos are not special

Consider

$(p(x)y')' + q(x)y + \lambda w(x) y = 0$ $\quad p, q, w > 0$

interval $[a, b]$

B.C. $\alpha y(a) + \beta y'(a) = 0$
$\gamma y(b) + \delta y'(b) = 0$

Morally equivalent to Sine & Cosine (& Fourier)

Famous Examples:

(1) $p(x) = x$; $q(x) = -\frac{\nu^2}{x}$; $w(x) = x$

$(xy')' - \frac{\nu^2}{x} y + \lambda x y = 0$

$xy'' + y' - \frac{\nu^2}{x} y + \lambda x y = 0$

$\boxed{x^2 y'' + x y' + (\lambda x^2 - \nu^2) y = 0}$ – Bessel's equation

(2) $(1-x^2) y'' - 2x y' + \lambda y = 0$

$\underbrace{((1-x^2)}_{p(x)} y')' + \lambda y = 0$

– Legendre's Equation

(3) $(e^{-x^2/2} y')' + \lambda e^{-x^2/2} y = 0$ Hermite equation

Basic Thm: (Greenberg, 17.7.1)

Let (ϕ_n, λ_n) be eigenfunctions & eigenvalues of S-L problem

Then:

(1) Eigenvalues are real

(2) ∞ number of eigenvalues $\lambda_1 < \lambda_2 < \dots < \lambda_n$, $\lambda_n \to \infty$ as $n \to \infty$

(3) If $\lambda_j \neq \lambda_k \Rightarrow \langle \phi_j, \phi_k \rangle = 0$

$$\langle \phi_j, \phi_k \rangle = \int_a^b dx\, w(x) \phi_j(x) \phi_k(x)$$

(4) $f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$

$a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}$

3/13/15 lecture 17

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x) \quad ; \quad a_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}$$

Before wednesday, ϕ_n $\sin\left(\frac{n\pi x}{\ell}\right), \cos\left(\frac{n\pi x}{\ell}\right)$

$$y'' = -\lambda^2 y \quad y(0) = y(\ell) = 0$$

$$M\vec{y} = -\lambda^2 \vec{y} \quad M = \begin{pmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & & 1 \end{pmatrix} \begin{matrix} \uparrow \\ N \\ \downarrow \end{matrix}$$

$\leftarrow N \rightarrow$

M $N > N$

N eigenvectors which are orthogonal $\vec{y}_n$

$$\vec{v} = \sum_n a_n \vec{y}_n$$

$$(\rho y')' + q(x) y + \lambda w(x) y = 0 \qquad \rho, w > 0$$

Outline for today

(1) prove orthogonality of eigenfunctions. ϕ_n

(2) Put equations in S-L form

(3) Bessel equations

Orthogonality proof

$\phi_n(x)$ $\phi_m(x)$

λ_n λ_m

$$\phi_n\left[(\rho \phi_n')' + q\phi_n + \lambda_n \phi_n w = 0\right]$$

$$-\phi_n\left[(\rho \phi_m')' + q\phi_m + \lambda_m \phi_m w = 0\right]$$

$$\int_a^b dx \left[\phi_n(\rho\phi_m')' - \phi_n(\rho\phi_n')'\right] + (\lambda_n - \lambda_m)\phi_n \phi_m w = 0$$

$$(\lambda_n - \lambda_m)\int_a^b \phi_n \phi_m w \, dx = 0$$

Putting eqns into S-L form

G 17.7 ext

$$(y'' - 2y' + \lambda y = 0)\sigma(x)$$

$y(0) = y(\pi) = 0$

$\sigma\, y'' - 2\sigma y' + \lambda \sigma y = 0$

SL $\underbrace{(py')'}_{py''+p'y'} + qy + \lambda w y = 0$

$p = \sigma \quad p' = -2\sigma \Rightarrow \sigma = p = e^{-2x}$

$$(e^{-2x}y')' + \lambda e^{-2x} y = 0$$

Bessel's Equation

$$(xy')' + \left(\lambda^2 x - \frac{\nu^2}{x}\right) y = 0$$

$$xy'' + y' + (\lambda^2 x - \frac{\nu^2}{x}) y = 0$$

$$\boxed{x^2 y'' + xy' + (\lambda^2 x^2 - \nu^2) y = 0}$$

(1) Resolve homick quandry
(2) Do an expansion

Behaviour near origin

You show that there is no solution of the form $y(x) = \sum_n a_n x^n$

Frobenius: $y(x) = x^s \sum_n a_n x^n$

$y = a_0 x^s + a_1 x^{s+1} + a_2 x^{s+2} + \dots$

$y' = s a_0 x^{s-1} + \dots$

$y'' = s(s-1) a_0 x^{s-2} + \dots$

$$a_0 s(s-1) x^s + a_0 s x^s + a_0 (-\nu^2) x^s$$

$$a_0 \underbrace{\left[s(s-1) + s - \nu^2 \right]}_{\text{indicial equation}} = 0$$

$$s(s-1) + s - \nu^2 = 0$$

$$s^2 - \nu^2 = 0$$

$$\boxed{s = \pm \nu}$$

Message

- Solutions exist (!)
- Solutions unruly
  - One of them diverges
  - The other has singular derivations

$0 \le x \le 1$

$f(x) = 1$

$f(x) = \sum_n a_n \phi_n(x)$ $\quad \alpha_n \to (n+\frac{1}{2})\pi + \frac{\pi}{4}$

Notes: $V = 0$ $\quad \phi_n(x) = J_0(\alpha_n x)$

$\sin(n\pi x)$

$$a_n = \frac{\int_0^1 dx\, f(x)\, J_0(\alpha_n x)}{\int_0^1 dx\, x\, J_0(\alpha_n x)^2}$$

3/23/15 lecture 18

Boundary value problems

$$y'' + \alpha^2 y = 0$$

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y + \lambda w(x) y = 0$$

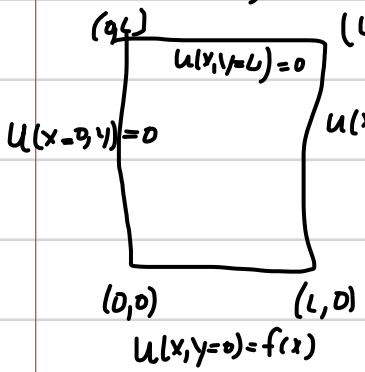
$$\langle \phi_n, \phi_m \rangle = \int_0^L dx\, w(x)\, \phi_m(x)\phi_n(x) = 0, \quad m \neq n$$

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$$

$$a_n = \frac{\langle f(x), \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}$$

Laplaces equation

$$y'' = 0, \quad y(0) = A, \quad y(1) = B$$



$$y = a + bx$$

$$a = A, \quad \begin{matrix} a+b = B \\ b = B - A \end{matrix}$$

$$y = A + (B-A)x$$

$$\nabla^2 u = 0 \qquad u(x,y)$$

$$\Delta u = 0$$

$$\nabla^2 u = \partial_{xx} u + \partial_{yy} u$$

Separation of variables

Guess $u(x,y) = \overline{X}(x)\overline{Y}(y)$

$$\nabla^2 u = \left[\overline{X}''(x)\overline{Y}(y) + \overline{Y}''(x)\overline{X}(x)\right] / \overline{X}(x)\overline{Y}(y)$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\underbrace{\frac{X''}{X}}_{-\alpha^2} + \underbrace{\frac{Y''}{Y}}_{\alpha^2}$$

$$-\alpha^2 + \alpha^2 = 0$$

$$\overline{X}'' + \alpha^2 \overline{X} = 0 \qquad \overline{X} = A\cos\alpha x + B\sin\alpha x$$

$$\overline{Y}'' - \alpha^2 \overline{Y} = 0 \qquad \overline{Y} = Ce^{\alpha y} + De^{-\alpha y}$$

$$u(x=L,y) = \overline{X}(L)\overline{Y}(y) = 0$$

$$\overline{X}'' + \alpha^2 \overline{X} = 0, \quad \overline{X}(0) = \overline{X}(L) = 0$$

$$A = 0, \quad B\sin\alpha L = 0 \rightarrow \alpha = \frac{n\pi}{L}$$

$$Y = Ce^{\alpha y} + De^{-\alpha y}$$

$$Y(L) = 0 \qquad Ce^{\alpha L} + De^{-\alpha L} = 0$$

$$\frac{C}{D} = -e^{-2\alpha L}$$

$$Y(y) = C\left(e^{\alpha y} - e^{2\alpha L - \alpha y}\right)$$

$$\alpha = \frac{n\pi}{L}$$

$$u = X(x)Y(y) = \left(B_n \sin\frac{n\pi x}{L}\right)\left(C\left(e^{\frac{n\pi}{L}y} - e^{2n\pi - \frac{n\pi}{L}y}\right)\right)$$

$$\nabla^2 u = 0$$

$$\nabla^2 u_n = 0$$

$$\nabla^2 u_m = 0$$

$$\nabla^2 (u_n + u_m) = 0$$

$$u = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi x}{L}\left(e^{\frac{n\pi}{L}y} - e^{2n\pi - \frac{n\pi}{L}y}\right)$$

$$u(x, y=0) = \sum_{n=0}^{\infty} A_n \sin\frac{n\pi x}{L}\left(1 - e^{2n\pi}\right) = f(x)$$

$$A_n = \frac{\langle f(x), \sin\frac{n\pi x}{L}\rangle}{\langle \sin\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\rangle} \frac{1}{1 - e^{2n\pi}}$$

$\nabla^2 u = 0$

Polar coordinates

$$\nabla^2 u = \frac{1}{r}\partial_r\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$u(R, \theta) = g(\theta)$$

$$u = R(r)\,\Theta(\theta)$$

$$\underbrace{\left[\frac{r(rR')'}{R}\right]}_{r} + \underbrace{\frac{\Theta''}{\Theta}}_{\theta} = 0$$

$$= \alpha^2 \qquad -\alpha^2 = 0$$

Helmholz equation → eigenvalue problem of this

$$\nabla^2 u = \lambda^2 u \qquad u|_{\text{Boundary}} = 0$$

$$u = XY, \quad X''Y + Y''X = \lambda^2 XY$$

$$\frac{X''}{X} + \frac{Y''}{Y} = \lambda^2$$

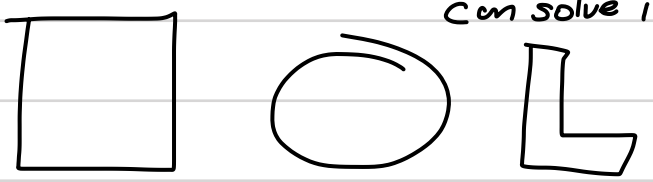
3/25/15 lecture 19

Partial differential equations

Last time: Laplace equation

$\nabla^2 u = \partial_{xx} u + \partial_{yy} u = 0$

can solve it on any shape



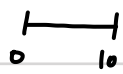
Idea of separation of variables $u(x,y) = X(x)\,Y(y)$

Big picture:

$\frac{du}{dt} = u^5 - 3u + t$

technical/detail — describe solutions

$\partial_t u = D\partial_{xx} u$ Diffusion equation

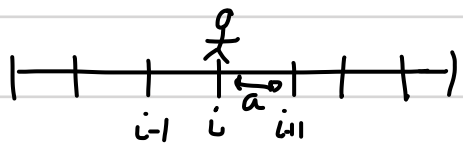


$u(t=0, x) = u_0(x)$

$u(t, x=0) = 5$

$u(t, x=10) = 7$

Random Walks



P_i^n = probability of being at site i at time n.

$P_i^{n+1} = \frac{1}{2} P_{i-1}^n + \frac{1}{2} P_{i+1}^n$

τ → time between steps

$t = n\tau$

$x = ia$

$P_i^n = P(\underbrace{ia}_{x}, \underbrace{n\tau}_{t})$

$P(ia, (n+1)\tau) = P(x, t+\tau)$

$P((i+1)a, n\tau) = P(x+a, t)$

$P(x, t+\tau) = \frac{1}{2}\left(P(x-a, t) + P(x+a, t)\right)$

Suppose $\tau << t$, $a << x$

$P(x, t+\tau) \approx P(x,t) + \tau \frac{\partial P}{\partial t} + \frac{\tau^2}{2} \frac{\partial^2 P}{\partial t^2} \cdots$

$P(x \pm a, t) = P(x,t) \pm a\partial_x P + \frac{a^2}{2}\partial_{xx} P + \cdots$

$P(x,t) + \tau \partial_t P + \frac{\tau^2}{2}\partial_{tt} P = \frac{1}{2}\left(P - a\partial_x P + \frac{a^2}{2}\partial_{xx} P + P + a\partial_x P + \frac{a^2}{2}\partial_{xx} P + \cdots\right)$

$= P + \frac{a^2}{2}\partial_{xx} P + \cdots$

$$\tau\, \partial_t P = \frac{a^2}{2} \partial_{xx} P$$

$$\boxed{\partial_t P = \underbrace{\frac{a^2}{2\tau}}_{D} \partial_{xx} P}$$

3/27/15 lecture 20

Not present

3/30/15 Lecture 21

Last class

$\partial_t u = D\partial_{xx} u$

$u(x,0) = \delta(x-x')$

$G = \frac{A}{\sqrt{Dt}} \exp\left(\frac{-(x-x')^2}{4Dt}\right)$

$\int_{-\infty}^{\infty} dx\, G(x-x',t) = 1 \qquad A = \frac{1}{\sqrt{4\pi}}$

$$\boxed{G(x-x',t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(\frac{-(x-x')^2}{4Dt}\right)}$$

Today: (1) What it is good for.

(2) What it is not good for.

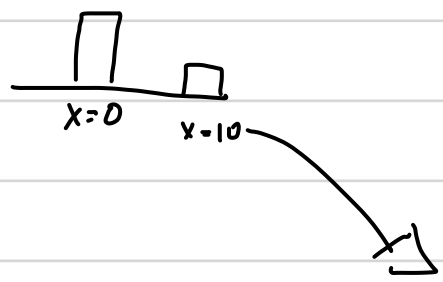
Example: Consider 100 random walkers located at x=0

and 10 " " " dt x=10

$a=1,\ \tau=\frac{1}{2} \Rightarrow D = \frac{a^2}{2\tau} = 1.$

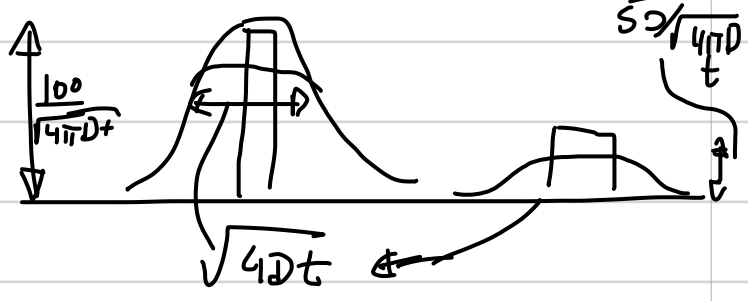
What happens?

$n(x,0) = 100\,\delta(x) + 50\,\delta(x-10)$



$n(x,t) = 100\,G(x,t) + 50\,G(x-10,t)$

$= \frac{100}{\sqrt{4\pi Dt}} \exp\left(\frac{-(x)^2}{4Dt}\right) + \frac{50}{\sqrt{4\pi Dt}} \exp\left(\frac{-(x-10)^2}{4Dt}\right).$



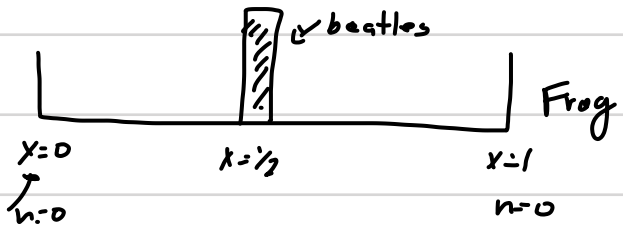
Two curves merge

$10 = \sqrt{4Dt} \Rightarrow t = \frac{100}{4D} = 25$

$n(x,t) = \int dx'\, n(x',0)\, G(x-x',t)$

$n(x,0) = \int dx'\, n(x',0)\, \delta(x-x')$

What it is not good for



$$\partial_t n = D\partial_{xx} n$$
$$n(x,0) = f(x)$$
$$n(0,t) = n(1,t) = 0$$

What to do?

Separation of Variables (Greenberg Sec. 18.3)

$$\partial_t n = D\partial_{xx} n$$
$$n(x,0) = f(x) \qquad \boxed{0 \le x \le 1}$$

Boundary Conditions $n(0,t) = 0$, $n(1,t) = 0$

Use ansatz $n(x,t) = X(x)\,T(t)$

$$\partial_t n = X(x)\,\frac{dT}{dt}$$
$$\partial_{xx} n = T(t)\,\frac{d^2X}{dx^2}$$
$$\frac{X\dot{T}}{XT} = \frac{DTX''}{XT}$$
$$\frac{\dot{T}}{T} = D\frac{X''}{X} = -D\alpha^2$$

D is positive

X-structure

$$X'' + \alpha^2 X = 0 \Rightarrow X(x) = A\cos\alpha x + B\sin\alpha x$$

T-structure

$$\dot{T} + D\alpha^2 T = 0 \Rightarrow T(t) = e^{-D\alpha^2 t}$$

$$n(x,t) = X(x)\,T(t)$$
$$n(0,t) = n(1,t) = 0 \Rightarrow X(0) = X(1) = 0$$
$$X(0) = A = 0$$
$$X(x) = B\sin\alpha x$$
$$X(1) = 0 \qquad \boxed{\alpha_n = n\pi}$$

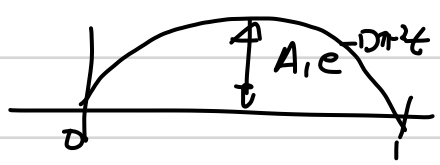
$$n(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, e^{-(n\pi)^2 Dt}$$

$$n(x,t=0) = \boxed{f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)}$$

$$A_n = \frac{\int_0^1 f(x)\sin(n\pi x)\,dx}{\int_0^1 \sin^2(n\pi x)\,dx}$$

$$h(x,t) = \boxed{A_1 \sin(\pi x) e^{-D\pi^2 t}} + A_2 \sin(2\pi x) e^{-4\pi^2 D t} + \dots$$

Long times



4/3/15 lecture 22

Diffusion:

$\partial_t u = D \partial_{xx} u$

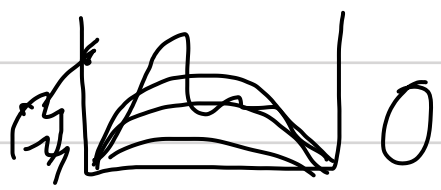
Initial for t: $u(x,0) = f(x)$

Boundary for x

$u(0,t) = u_1$, $u(L,t) = u_2$

$u'(0,t) = u'(L,t) = 0$

Frogs / Flies

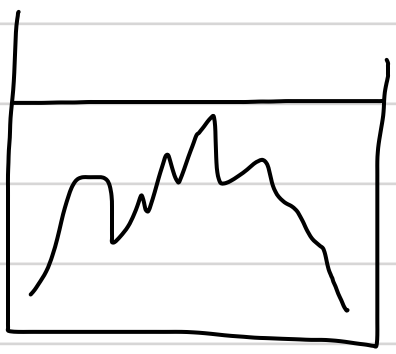


Leading term

$u(x,t) \sim A_1 e^{-\frac{\pi^2}{L^2} D t} \sin\left(\frac{\pi x}{L}\right)$

No flux boundary

$u'(0,t) = u'(L,t) = 0$

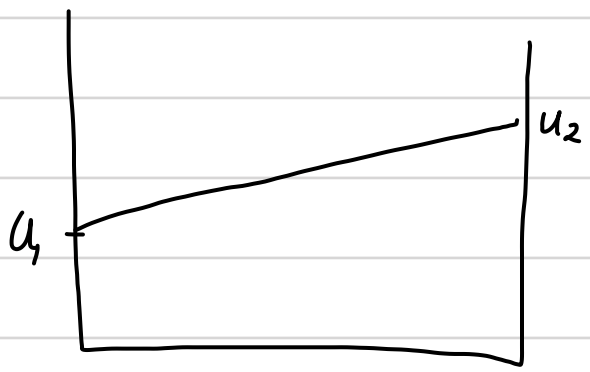


$u(x, t \to \infty) \sim B_n e^{-n^2 t} \cos\left(\frac{\pi n x}{L}\right)$

Inhomogeneous Boundaries

$u(0,t) = u_1$, $u(L,t) = u_2$, As $t \to \infty$

$u(x,t) = A + Bx = \boxed{u_1 + (u_2 - u_1)\frac{x}{L}}$



$u(x,t) = u_s(x) + \bar{u}(x,t)$

$u(x,t=0) = f(x) = u_s(x,0) + \bar{u}(x,t=0) = A + Bx$

$\bar{u}(x,0) = f(x) - (A + Bx)$

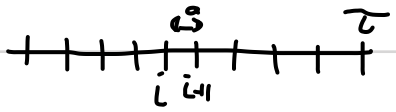
$u(0,t) = u_1 = u_s(0,t) + \bar{u}(0,t)$

$\hookrightarrow u_1$

$\bar{u}(0,t) = 0$

$\bar{u}(L,t) = 0$

Review: Random Walkers



time $t+\tau$ probability for being at site i, $P_i^{t+\tau}$

$$P_i^{t+\tau} = \frac{1}{2}\left(P_{i+1}^t + P_{i-1}^t\right)$$

$$P(x,t+\tau) = \frac{1}{2}\left(P(x+a,t) + P(x-a,t)\right)$$

$$\partial_t P(x,t) = \underbrace{\frac{a^2}{2\tau}}_{D} \partial_{xx} P(x,t)$$

Right Mover

$$P_i^{t+1} = P_{i-1}^t$$

$$R(x,t+\tau) = R(x-a,t) \leftarrow \text{Right movers } R(x,t)$$

Left movers $L(x,t)$

$$L(x,t+\tau) = L(x+a,t)$$

$$R(x,t+\tau) = R(x,t) + \tau \partial_t R(x,t) + \dots$$

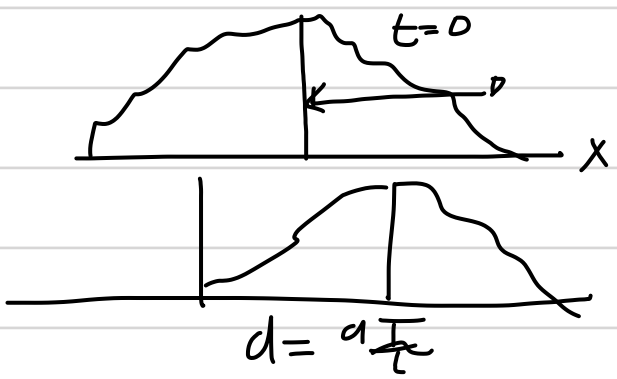
$$R(x-a,t) = R(x,t) - a\,\partial_x R(x,t) + \dots$$

$$\boxed{\partial_t R = -\frac{a}{\tau}\partial_x R \qquad \partial_t L = \frac{a}{\tau}\partial_x L}$$

Advection Equations

$R(x,t)$

$t=0$ $R(x,0) = f(x)$



Solution to $\partial_t R = \frac{a}{\tau} d_x R$

$R(x,0) = f(x)$

$$R(x,t) = f\left(x - \frac{a}{\tau}t\right)$$

$$\partial_t R = f'\left(x - \frac{a}{\tau}t\right) \times \left(\frac{-a}{\tau}\right)$$

$$\underbrace{\frac{-a}{\tau}}_{\text{velocity}} \partial_x R = f'\left(x - \frac{a}{\tau}t\right) \times \left(-\frac{a}{\tau}\right)$$

L.H.S = R.H.S

$\frac{1}{2}$ L movers & $\frac{1}{2}$ Right Movers

$$n(x,t) = R(x,t) + L(x,t)$$

$$\partial_t n = \partial_t R + \partial_t L = -\frac{a}{c}\partial_x R + \frac{a}{c}\partial_x L$$

$$= -\frac{a}{c}\partial_x (R-L)$$

$$\partial_t \left[\partial_t n = -\frac{a}{c}\partial_x (R-L)\right]$$

$$\partial_{tt} n = -\frac{a}{c}\partial_x (\partial_t R - \partial_t L)$$

$$\partial_{tt} n = \frac{a}{c}\partial_x \left(-\frac{a}{c}\partial_x R - \frac{a}{c}\partial_x L\right)$$

$$\partial_{tt} n = \frac{a^2}{c^2}(\partial_{xx} R + \partial_{xx} L) = \frac{a^2}{c^2} 2\partial_{xx} n$$

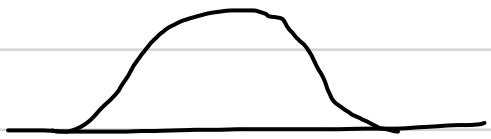
Wave Equation

$$\partial_{tt} n = c^2 \partial_{xx} n$$

4/6/15 Lecture 23

Today
- Differences between Disciplined vs. Random Walkers
- Separation of variables
  - Inhomogeneous BC (Green's 18.3)

Work equation ↔ Musical Notes

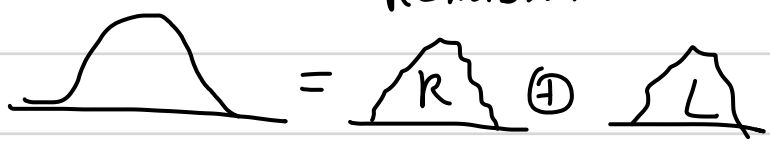


Is it the same

50% R or 50% L

vs

Random



$\partial_t u = D\partial_{xx} u$

$\partial_{tt} u = c^2 \partial_{xx} u \quad u = L + R$

$\partial_t R = -V \partial_x R \quad V = \frac{a}{\tau}$

$\partial_t L = V \partial_x L$

d'Alambert's Solution

$R(x,t) = f(x - Vt)$

$L(x,t) = g(x + Vt)$

$\partial_t R = \frac{df}{dv}\frac{dv}{dt} = -V\frac{df}{dy} = -V\partial_x R$

$\partial_t R = \frac{df}{dy}\frac{dv}{dx} = \frac{df}{dy}$

$R(x,0) = f(x)$

Inhomogeneous B.C.

$\partial_t u = D\partial_{xx} u \qquad u(0,t) = u_1 \qquad u(L,t) = u_2$ ← inhomogeneous

$u(x, t=0) = f(x)$

Recall: $u(x,t) = \overline{X(x)T(t)}$

$u(x,t) = u_{steady}(x) + \bar{u}(x,t)$

↑ takes care of inhomog. B.C.

$\partial_t u_{steady} = 0$

$\frac{\partial^2 u_{steady}}{\partial x^2} = 0 \rightarrow u_{steady} = A + Bx$

$u_{steady}(0) = u_1$

$u_{steady}(L) = u_2$

$$\boxed{u_{steady} = u_1 + \left(\frac{u_2 - u_1}{L}\right)x}$$

$$\partial_t(u_s + \bar{u}) = D\partial_{xx}(u_s + \bar{u})$$

$$\partial_t \bar{u} = \cancel{D\partial_{xx}u_s'} + D\partial_{xx}\bar{u}$$

$$u(0,t) = u_1 = \underbrace{u_s(0)}_{u_1} + \bar{u}(0,t) \Rightarrow \boxed{\bar{u}(0,t) = 0}$$

$$u(L,t) = u_2 = \underbrace{u_s(L)}_{u_L} + \bar{u}(L,t) \Rightarrow \boxed{\bar{u}(0,L) = 0}$$

I.C.

$$u(x,0) = f(x) = u_s(x) = \bar{u}(x,0)$$

$$\boxed{\bar{u}(x,0) = f(x) - u_s(x)}$$

$$\partial_{tt}u = c^2\partial_{xx}u$$

↑ depends on tension density etc

$u(x,0) = f(x)$ $\quad u(x,t) = ?$

$u(0,t) = 0$ $\quad u(L,t) = 0$

Inhomogeneous B.C.

$\partial_{tt}u = c^2\partial_{xx}u$ $\quad u(0,t) = 0$ $\quad u(L,t) = 0$

$u(x,t=0) = f(x)$ $\quad \dot{u}(x,t=0) = 0$

Recall: $u(x,t) = X(x)T(t)$

$$\frac{X\frac{\partial^2 T}{\partial t^2}}{XT} = c^2\frac{\frac{\partial^2 X}{\partial x^2}T}{XT} = -k^2$$

$$\frac{\partial^2 T}{\partial t^2} + k^2 T = 0 \qquad T(t) = A\cos(kt) + B\sin(kt)$$

$$\frac{\partial^2 X}{\partial x^2} + \frac{k^2}{c^2}X = 0 \qquad X(t) = C\cos\left(\frac{kx}{c}\right) + D\sin\left(\frac{kx}{L}\right)$$

B.C $u(0) = u(L) = 0$

$X(0) = X(L) = 0$

$\downarrow$

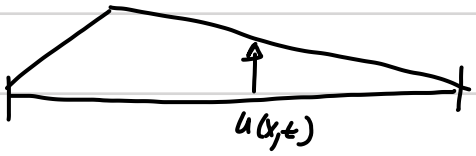
$C = 0$

$X(x) = D\sin\left(\frac{k}{c}x\right)$

$X(L) = 0 = D\sin\left(\frac{kx}{c}\right)$

$\frac{kL}{c} = n\pi$ $\quad$ $k = \frac{n\pi c}{L}$

4/8/15 lecture 24



$u(x,0) = f(x)$
$u(0,t) = u(L,t) = 0$

$$\partial_{tt} u = c^2 \partial_{xx} u$$

$$u(x,t) = X(x)\,T(t)$$

$$\frac{\ddot{T}X}{XT} = c^2 \frac{X''T}{XT} = -\alpha^2$$

B.C. $u(0,t) = 0 = X(0)T(t) \Rightarrow X(0) = 0 = c$

$u(L,t) = 0 \quad X(L) = 0 = D\sin\left(\frac{\alpha L}{c}\right)$

$$T(t) = A\cos\alpha t + B\sin\alpha t$$

$$X(x) = C\cos\left(\frac{\alpha x}{c}\right) + D\sin\left(\frac{\alpha x}{c}\right)$$

$$\boxed{\frac{\alpha L}{c} = n\pi}$$

Most General Solution

$$u(x,t) = \sum_{n=1} \left(A_n \cos\alpha_n t + B_n \sin\alpha_n t\right) \sin\left(\frac{\alpha_n x}{c}\right)$$

I.C. $u(x,0) = f(x) = \boxed{\sum_n A_n \sin\frac{\alpha_n x}{L}} = \boxed{\sum_n A_n \sin\frac{n\pi x}{L}}$

$$\partial_t u(x,t) = \sum_{n=1}^{\infty} \left(-A_n \alpha_n \sin(\alpha_n t) + B_n \alpha_n \cos(\alpha_n t)\right) \sin\frac{\alpha_n x}{L}$$

$$\partial_t u(x,0) = 0 = \boxed{\sum_n \alpha_n B_n \sin\frac{\alpha_n x}{c}} = \sum_n \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right)$$

If I required $\partial_t u(x,0) = g(x)$

$$g(x) = \sum_n \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right)$$

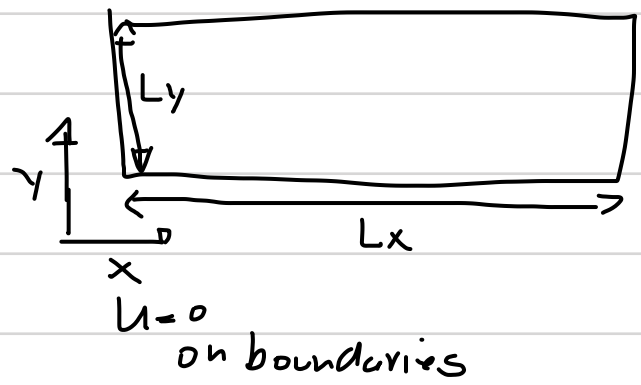
$$B_n = \frac{\int_0 g(x) \sin\left(\frac{n\pi x}{L}\right) dx}{\int_0 \sin^2\left(\frac{n\pi x}{L}\right) dx}$$

$$u(x,t) = \sum_n A_n \cos\frac{n\pi c}{L} t \sin\left(\frac{n\pi x}{L}\right)$$

$$= A_1 \cos\left(\frac{\pi c}{L} t\right) \sin\left(\frac{\pi x}{L}\right) + A_2 \cos\left(\frac{2\pi c}{L} t\right) \sin\left(\frac{2\pi x}{L}\right) + \dots$$

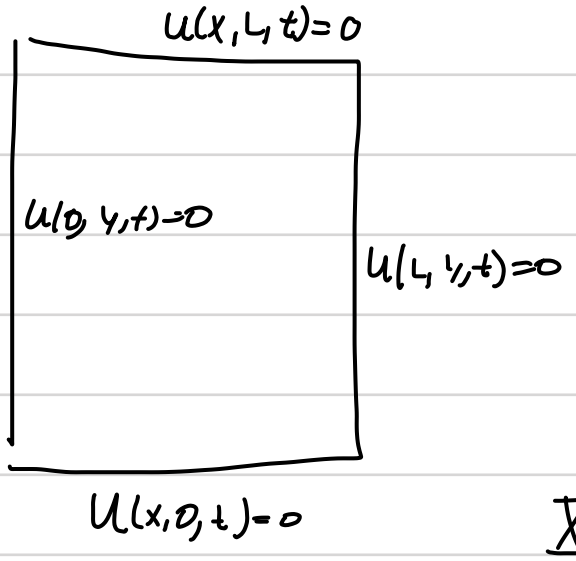
2D Wave Equation

$$\partial_{tt} u = c^2 \nabla^2 u = c^2 (\partial_{xx} u + \partial_{yy} u)$$



$$u(x, y, t=0) = f(x, y)$$

$$\partial_t u(x, y, t=0) = 0$$



Separation of variables

$$u(x, y, t) = X(x) Y(y) T(t)$$

$$\left[\frac{\partial^2 T}{\partial t^2} XY = c^2 \left(\frac{\partial^2 X}{\partial x^2} YT + \frac{\partial^2 Y}{\partial y^2} XT \right) \right] \frac{1}{XYT}$$

$$\frac{X''}{X} = -\frac{\alpha^2}{c^2} \qquad \frac{Y''}{Y} = -\frac{\beta^2}{c^2}$$

$$\frac{\ddot{T}}{T} = c^2 \left(-\frac{\alpha^2}{c^2} - \frac{\beta^2}{c^2} \right) = -(\alpha^2 + \beta^2)$$

Most General Solution

$$X(x) = A \sin\left(\frac{\alpha}{c} x\right) + B \cos\left(\frac{\alpha}{c} x\right)$$

$$Y(y) = C \sin\left(\frac{\beta}{c} y\right) + D \cos\left(\frac{\beta}{c} y\right)$$

$$T(t) = E \sin\left(\sqrt{\alpha^2+\beta^2}\, t\right) + F \cos\left(\sqrt{\alpha^2+\beta^2}\, t\right)$$

B.C. $u(0, y, t) = 0 = X(0) Y(y) T(t)$

$$X(0) = 0 \Rightarrow B = 0$$

$$u(L_x, y, t) = 0 \quad X(L_x) Y(y) T(t)$$

$$X(L_x) = 0 \Rightarrow \boxed{\frac{\alpha L_x}{c} = n\pi}$$

Other two B.C.

$u(x,0,t)=0 \Rightarrow D=0$

$u(x,L_y,t)=0 \Rightarrow \boxed{\frac{\beta_n L_y}{c} = n\pi}$

$\alpha_n = \frac{n\pi c}{L_x} \qquad \beta_n = \frac{n\pi c}{L_y}$

Most general solution

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[E_{nm}\sin\left(\sqrt{\alpha_n^2+\beta_n^2}\,t\right) + F_{nm}\cos\left(\sqrt{\alpha_n^2+\beta_n^2}\,t\right)\right]\sin\frac{\alpha_n x}{c}\sin\frac{\beta_m}{c}y$$

$$\text{Modes} = \sqrt{\alpha_n^2+\beta_n^2} = \pi c\sqrt{\frac{n^2}{L_x^2}+\frac{m^2}{L_y^2}}$$

4/10/15 Lecture 25

Wave equation

→ $\partial_{tt} u = c^2 \partial_{xx} u$ $\qquad \partial_{tt} u = c^2 \nabla^2 u$

→ Rectangle

$$u(\vec{x},t) = e^{i\omega t} F(\vec{x})$$

$$-\omega^2 F = c^2 \nabla^2 F \Rightarrow \nabla^2 F = -\frac{\omega^2}{c^2} F$$

$$\boxed{\nabla^2 F = -\lambda^2 F}$$

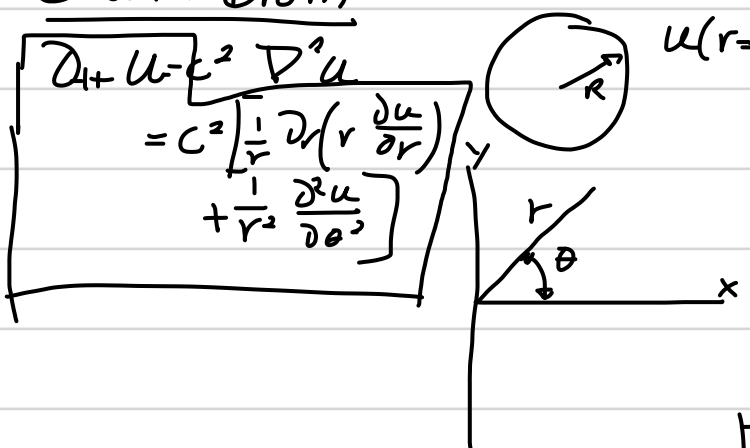
$\nabla^2 F = -\lambda^2 F$ Helmholtz Eqn.

$F(\text{boundary}) = 0$

(ii) Circle

(i) Nonlinear

Circular Drum



$u(r=R,\theta,t)$

$$\partial_{tt} u = c^2 \nabla^2 u = c^2 \left[\frac{1}{r}\partial_r\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right]$$

$$u(r,\theta,t) = T(t)\,R(t)\,\Theta(\theta)$$

$$\frac{\ddot{T}R\Theta}{TR\Theta} = c^2\left(\frac{I\Theta\left(\frac{1}{r}(rR')'\right)}{TR\Theta} + \frac{1}{r^2}\frac{\Theta'' RT}{TR\Theta}\right) = -k^2$$

$$\boxed{\frac{\partial^2 T}{\partial t^2} = -k^2 T} \qquad \left[\frac{R'' + \frac{1}{r}R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\frac{k^2}{c^2}\right] r^2$$

$$\underbrace{r^2\left[\frac{R'' + R'/r}{R}\right] + \frac{k^2}{c^2}r^2}_{\text{Depends on } r} = \underbrace{-\frac{\Theta''}{\Theta}}_{\text{depends on } \Theta} = \alpha^2$$

$$\boxed{\Theta'' + \alpha^2 \Theta = 0}$$

$$\frac{r^2 R'' + rR'}{R} + \frac{k^2 r^2}{c^2} = \alpha^2$$

$$\boxed{r^2 R'' + rR' + \left(\frac{k^2 r^2}{c^2} - \alpha^2\right) R = 0}$$

$\Theta(\theta) = A\cos\alpha\theta + B\sin(\alpha\theta)$

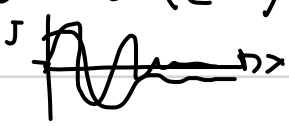
Must be periodic in theta with period 2π

$\boxed{\alpha = n}$

$$\boxed{R'' + \frac{1}{r}R' + \left(\frac{k^2}{c^2} - \frac{n^2}{r^2}\right)R = 0}$$

Bessels equation

$$R(r) = C\,J_n\left(\frac{k}{c}r\right) + D\,\cancel{Y_n\left(\frac{k}{c}r\right)}$$

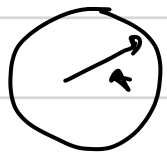


Diverges at origin

X_{nm}

n^{th} Bessel

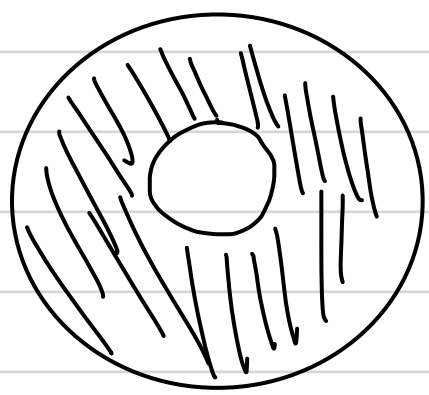
m^{th}



$R(A) = 0$

$= C J_n\left(\frac{k}{c}A\right) = 0$

$\frac{k}{c}A = X_{nm} \Rightarrow \boxed{K_{nm} = X_{nm}\frac{C}{A}}$



Anulus

4/13/15 Lecture 26

We have studied:

$\partial_t u = D\partial_{xx} u$ diffusion

$\partial_t u = V\partial_x u$

$\partial_t u = V\partial_1 u$

ODE's

$\frac{du}{dt} = u$

$\frac{du}{dt} = u^3 + \dots$

$$\partial_t u = D\partial_{xx} u^5 - u^3 - V\frac{\partial_x u}{1+u}$$

$$\partial_t \vec{u} + \overbrace{\boxed{\vec{u}\cdot\nabla\vec{u}}}^{\text{advection}} = -\nabla p + \overbrace{\boxed{\nabla^2\vec{u}}}^{\text{diffusion}}$$

$$\nabla\cdot\vec{u} = 0$$

$$c\,\partial_t u = +\partial_x f + s$$

$f = u$

$c = 1$

$\partial_t u = \partial_x u$

$u = f(\overbrace{x+t}^{r})$

Advection + ODE

$\partial_t u = V\partial_x u + f(u,t)$

Advection $\partial_t u = -V_{(x,t)}\partial_x u \rightarrow u = g(x - Vt)$

& speed of disciplined walker

$u(x,t) = G(x - X_0(t)) \rightarrow \partial_t u = \frac{dG}{dy}\frac{dy}{dt} = \frac{dG}{dy}\left|-\frac{dx_0}{dt}\right|$

$$\partial_x u = \frac{dG}{dy}\frac{dy}{dx} = \frac{dG}{dy}$$

$$\partial_t u = -\frac{dx_0(t)}{dt}\frac{dG}{dy} = -\frac{dx_0}{dt}\partial_x u \qquad \boxed{\frac{dx_0}{dt} = V(x_0, t, u(x_0))}$$

If V is constant

$x_0 = Vt$

$$\frac{d\,u(x(t),t)}{dt} = \partial_t u + \frac{dx}{dt}\partial_x u = f(u,t)$$

$$\boxed{\frac{du}{dt} = f(u,t)}$$

Ex.1

$$\partial_t u + 5\partial_x u = 3$$

$$u(x,0) = \cos x$$

$u(x(t),t)$

$$\frac{dx}{dt} = 5 \qquad \frac{du}{dt} = 3 \rightarrow u = 3t + A(x_0)$$

$$x = x_0 + 5t \qquad u(x_0,0) = \cos x_0 = A(x_0)$$

$$x_0 = x - 5t \qquad u(x,t) = \cos(x_0) + 3t$$

$$\boxed{u(x,t) = \cos(x-5t) + 3t}$$

Sidenote

$$\partial_t u + a(x,t)\partial_x u = g(x,t)$$

$$\frac{dx}{dt} = a(x,t) \qquad u(r,t) = u(x(t),t)$$

$$\frac{dr}{dt} = g(r,t)$$

Ex.2

$$\partial_t u + 3\partial_x u = u$$

$$\frac{dx}{dt} = 3 \rightarrow x = x_0 + 3t$$

$$x_0 = x - 3t$$

$$\frac{du}{dt} = u \qquad u(t) = A(x_0)e^t$$

$$u(t=0) = A(x_0) = f(x_0)$$

$$\boxed{u(x,t) = f(x_0)e^t = f(x-3t)e^t}$$

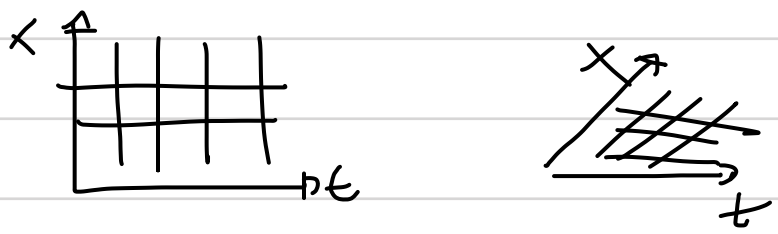
4/15/15 Lecture 27

Method of Characteristics

$\partial_t u + a(x,t,u)\partial_x u = b(x,t,u)$

$u(x,t=0) = f(x)$

$u_x(x,t) = u(\bar{x}(s), \bar{t}(s))$



$$\frac{du}{ds} = \partial_x u \frac{dx}{ds} + \partial_t u \frac{dt}{ds}$$

$\frac{dx}{ds} = a(x,t,u)$

$\frac{dt}{ds} = 1$

$\frac{du}{ds} = b(x,t,u)$ start → $u(x,t=0) = f(x)$

rewrit → $u(x(s=0), t(s=0)) = f(x(s=0))$

$x_0 \quad t_0$

$\frac{dx}{ds} = a(x,t,u)$ $\qquad u(x_0,0) = f(x_0)$

$X(s=0) = x_0$

$$\boxed{\frac{dx}{dt} = a(x,t), \quad x(s=0) = x_0}$$

$$\boxed{\frac{du}{ds} = b(x,t,u), \quad u(x_0,0) = f(x_0)}$$

Advection

$\partial_t u + V\partial_x u = 0, \quad u(x,0) = f(x)$

$u(x,t) = f(x - Vt)$

$\frac{dx}{dt} = V, \quad \frac{du}{dt} = 0$

$X(t=0) = x_0 \quad u(t=0) = f(x_0)$ $\qquad X = Vt + x_0 \rightarrow x_0 = X - Vt$

$u = u_0 = f(x_0)$

$$\boxed{u(x,t) = f(x - Vt)}$$

$\partial_t u + 5\partial_x u = 3,\ u(x,0) = \cos(x)$

$\frac{dx}{dt} = 5,\ x(t=0) = x_0 \qquad x = 5t + x_0$

$\frac{du}{dt} = 3,\ u(t=0) = \cos(x_0)$

$u = 3t + u_0 = 3t + \cos(y_0)$

$\boxed{u(x,t) = 3t + \cos(x - 5t)}$

$\partial_t u + 3\partial_x u = u \qquad u(x,0) = 2x$

$\frac{dx}{dt} = 3,\ x(0) = x_0 \qquad x = 3t + x_0$

$x_0 = x - 3t$

$\frac{du}{dt} = u \rightarrow u = u_0 e^t,\ u_0 = 2x_0$

$u(x,t) = 2(x - 3t)e^t$

Example:

$x\partial_t u - t\partial_x u = 0,\ u(x,0) = e^{-x^2}$

$\partial_t u - \frac{t}{x}\partial_x u = 0$

$\frac{dx}{dt} = \frac{-t}{x} \rightarrow \int x\,dx = \int t\,dt$

$x^2 - x_0^2 = -t^2$

$x_0 = \sqrt{x^2 - t^2} \qquad \frac{du}{dt} = 0 \qquad u = u_0 = f(x_0) = e^{-x_0^2}$

$= e^{-(x^2+t^2)}$

04/17/15 lecture 28

Not present- no notes

4/20/15 lecture 29

Midterm 3 topics

Separation of variables

$\partial_t u = \partial_{xx} u$
$\partial_t u = D\partial_{xx} u$
$\nabla u = \nabla^2 u - ku$

} homogeneous & inhomogeneous b.c.

- Random walkers
  - → diffusion
  - Discipled walkers → advection

- Greens Function method for diffusion (advection)

- Dynamics
  - Diffusion + ODE
  - Advection & ODE

  Advection / diffusion & Characteristics

$\partial_t u = D\partial_{xx} u \pm \alpha u$

$\partial_t u = D\partial_{xx} u + \alpha u(1-u)$
Fisher equation

Waves traveling @ $c = \sqrt{4\alpha D}$

$u(x,t) = F(\underbrace{x - ct}_{y})$ $\quad F' = \frac{dF}{dy}$

$\partial_t u = -cF'$

$\partial_{xx} u = F''$

$$\boxed{-cF' = DF'' + \alpha F(1-F)}$$

Method of Similarity Solutions

$u(x,t) = X(x)T(t)$

$$\boxed{u(x,t) = \bar{t}^{\,a}\, \bar{u}(\bar{x}\bar{t}^{\,b})}$$ today

non dimensional

$u \sim e^{-x^2/t}$ $\qquad \partial_t u = D\partial_{xx} u$, $u(x,0) = u_{IC}$, $u(0,t) = u_{B.C.}$

$$\bar{x} = \frac{x}{x_0}, \quad \bar{t} = \frac{t}{t_0}, \quad \bar{u} = \frac{u - u_{ic}}{u_0}$$

$$\frac{d\bar{u}}{d\bar{t}} = \frac{1}{u_0} \frac{\partial u}{\partial t} \frac{dt}{d\bar{t}} = \frac{t_0}{a_0} \frac{du}{dt}$$

$$\frac{d\bar{u}}{d\bar{x}} = \frac{x_0}{u_0} \frac{\partial u}{\partial x}, \quad \partial_{\bar{x}\bar{x}} \bar{u} = \frac{x_0^2}{u_0} \frac{\partial^2 u}{\partial x^2}$$

$$\frac{u_0}{t_0} \frac{\partial \bar{u}}{\partial \bar{t}} = D \frac{u_0}{x_0^2} \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}$$

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \underbrace{D \frac{t_0}{x_0^2}}_{\text{Dimensionless}} \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}$$

$$C = D \frac{t_0}{x_0^2}$$

$$x_0 t_0^b \Rightarrow \frac{x_0}{\sqrt{C D t_0}} \Rightarrow \text{dimensionless}$$

$$\downarrow C = 4$$

$$\boxed{\bar{u}\left(\frac{\bar{x}}{\sqrt{4D\bar{t}}}\right)}$$

$$\eta$$

$$\partial_t \bar{u} = \frac{x}{\sqrt{4D}} \left(-\frac{1}{2} \frac{1}{t^{3/2}}\right) \frac{d\bar{u}}{d\eta} = -\frac{1}{2} \frac{\eta}{t} \frac{d\bar{u}}{d\eta}$$

$$\partial_x \bar{u} = \frac{1}{\sqrt{4Dt}} \frac{d\bar{u}}{d\eta} \Rightarrow \partial_{xx} u = \frac{1}{4Dt} \frac{\partial^2 \bar{u}}{\partial \eta^2}$$

$$-\frac{1}{2} \frac{\eta}{t} \frac{\partial \bar{u}}{\partial \eta} = D \left(\frac{1}{4Dt}\right) \frac{\partial^2 u}{\partial \eta^2}$$

$$\boxed{\frac{\partial^2 \bar{u}}{\partial \eta^2} + 2\eta \frac{\partial \bar{u}}{\partial \eta} = 0} \quad u(x,t) = \bar{u}\left(\frac{x}{\sqrt{4Dt}}\right)$$

$$\partial_t u = \partial_{xx} u + R(u)$$

$$R(u) \sim u^p, \; p > 1$$

$$\frac{\partial u}{\partial t} = u^p$$

$$\text{Ansatz}: \; u(x,t) = t^a \bar{u}\left(\frac{x}{\sqrt{t}}\right)$$

$$\partial_t u = a t^{a-1} \bar{u} + t^a \left(\frac{-x}{2} \frac{1}{t^{3/2}}\right) \bar{u}'\left(\frac{x}{\sqrt{t}}\right)$$

$$\partial_x u = \frac{t^a}{\sqrt{t}} \bar{u}'\left(\frac{x}{\sqrt{t}}\right), \quad \partial_{xx} u = \frac{t^a}{t} \bar{u}''\left(\frac{x}{\sqrt{t}}\right)$$

$$\partial_t u = D \partial_{xx} u + u^p$$

$$t^{a-1}\left[a\bar{u} - \frac{x}{2\sqrt{t}} \bar{u}'\right] = t^{a-1} \bar{u}''\left(\frac{x}{\sqrt{t}}\right) + \left(t^a \bar{u}\left(\frac{x}{\sqrt{t}}\right)\right)^p$$

$$a\bar{u} - \frac{n}{2}\bar{u}' = \bar{u}'' + \underbrace{t^{ap-a+1}}\bar{u}^p$$

$$\Rightarrow \quad ap - a + 1 = 0 \quad \Rightarrow a = \frac{-1}{p-1}$$

$$u(x,t) = t^{\frac{1}{p-1}}\,\bar{u}\left(\frac{x}{\sqrt{t}}\right)$$

4/22/15 lecture 30

Map of Nonlinear PDE's

1 space dimension

$u(x,t)$

$$\partial_t u + \underbrace{V(x,t,u)\partial_x u}_{\text{advection}} = \underbrace{D\partial_{xx} u}_{\text{diffusion}} + \underbrace{S(x,t,u)}_{\text{ODE}}$$

Friday

$\partial_t u = D\partial_{xx} u + \alpha u(1-u)$

$u(x,t) = F(x - Vt)$

$V = \sqrt{4D\alpha}$

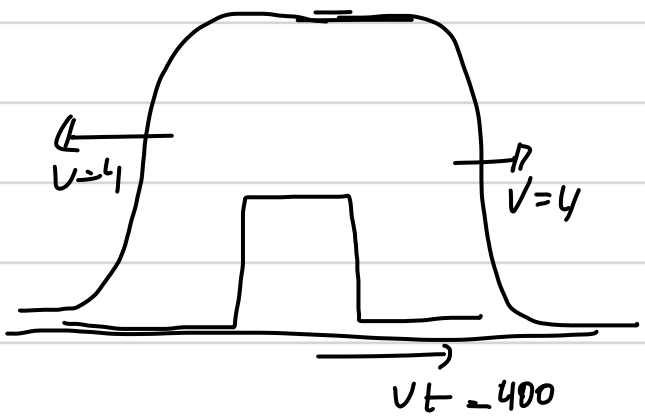
$$u(x,t) = \frac{1}{(t^*-t)} F\left(\frac{x}{\sqrt{t^*-t}}\right)$$

$\partial_t u = 4\partial_{xx} u + u(1-u)$

$u(x,0) = \frac{1}{2} \quad |x| < 1/2$

Describe solution at $t = 100$?

$V = \sqrt{4 \times D \times \alpha} = 4$



$\frac{du}{dt} = u \quad u = Ae^t \quad u(0) = 1/2 \quad u(t) = \frac{1}{2}e^{t-1}$

Example

$$\partial_t u = (5 + e^{-t})\partial_{xx} u + \left(3 + \frac{2}{1+t}\right)u - u^2$$

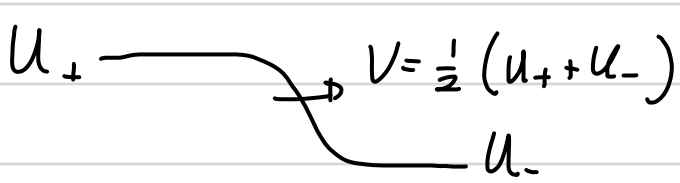
$u(x,0) = 0.1$
$|x| < 1$

$3u - u^2$

fixed points $u = 0$

$3 = u \rightarrow u = 3$

$\sqrt{4 \times 3 \times 5} = \sqrt{60} \sim 7.5$

Shocks

$\partial_t u + u\,\partial_x u = D\partial_{xx} u$