APMTH-105 Notes Section #5

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March 9, 2015

Goals for the week

1. Learn to use a power series approach to solve ODE's with variable coefficients.
2. Learn how to solve an eigenvalue problem.
3. Learn how to expand a function in a Fourier series.

Problem 1: Finding power series solution of first order ODE

From: Fundamental of Diff. Eq., Nagel 2012

Find a power series solution about $x = 0$ to

$$y' + 2xy = 0 \tag{1}$$

Solution

The coefficient of y is the polynomial 2x, which is analytic everywhere, so $x = 0$ is an ordinary[1] point of equation 1. Thus, we expect to find a power series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2}$$

where our task is to determine the coefficients a_n. For this purpose we need the expansion for $y'(x)$ that is given by termwise differentiation of (2):

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \tag{3}$$

We now substitute the series expansions for y an y' into (1) and obtain:

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n = 0 \tag{4}$$

which simplifies to

$$\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} 2 a_n x^{n+1} = 0 \tag{5}$$

To add the two power series in 5, we add the coefficients of like powers of x. If we write out the first few terms of these summations and add, we get

$$a_1 + (2a_2 + 2a_0)x + (3a_3 + 2a_1)x^2 + (4a_4 + 2a_2)x^3 + ... = 0 \tag{6}$$

So we can get the system of equations:

[1] By an ordinary point of a first-order equation $y' + q(x)y = 0$, we mean a point where $q(x)$ is analytic.

$$a_1 = 0, \quad 2a_2 + 2a_0 = 0,$$
$$3a_3 + 2a_1 = 0, \quad 4a_4 + 2a_2 = 0, \quad etc. \tag{7}$$

Solving the preceding system, we find

$$a_1 = 0, \quad a_2 = -a_0, \quad a_3 = -\tfrac{2}{3}a_1 = 0,$$
$$a_4 = -\tfrac{1}{2}a_2 = -\tfrac{1}{2}(-a_0) = \tfrac{1}{2}a_0 \tag{8}$$

Hence, the power series for the solution takes the form

$$\boxed{y(x) = a_0 - a_0x^2 + \frac{1}{2}a_0x^4 + ..}$$

Problem 2: Second order power sereies solution to ODE

From: AM105 P-Set 5 2013

Find the power series solution to the equation

$$\frac{d^2y}{dx^2} + \frac{3}{x-4}\frac{dy}{dx} + 4y = 0, \tag{9}$$

starting from the initial condition $y(0) = 3$ and $y'(0) = 0$. Find the radius of convergence of the series.

Solution

We will use

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \tag{10}$$

and substitute it into the original equation.
First, note that the coefficient function of $\frac{dy}{dx}$ in (9) depends on x and therefore we must expand it in a power series. Alternatively (and more simply) we can also multiply (9) through by $x-4$ and then notice that in the resulting equation

$$(x-4)\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + (4x-16)y = 0 \tag{11}$$

all of the coefficients are power series (of the functions $x-4$ and $4x-16$ around the point $x=0$).
Now, we will use $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and substitute into our new equation (11). Taking derivatives of $y(x)$ we have

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$$
$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

and substituting into (11) we get

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-1} - 4\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} +$$
$$3\sum_{n=1}^{\infty} a_n n x^{n-1} + 4\sum_{n=0}^{\infty} a_n x^{n+1} - 16\sum_{n=0}^{\infty} a_n x^n = 0 \tag{12}$$

Using the appropriate substitutions, namely $p = n - 2$, $m = n - 1$ and $q = n + 1$ and defining $a_{-1} = 0$ we can rewrite this as

$$\sum_{m=1}^{\infty} a_{m+1}(m+1)mx^m - 4\sum_{p=0}^{\infty} a_{p+2}(p+2)(p+1)x^p + \\ 3\sum_{m=0}^{\infty} a_{m+1}(m+1)x^m + 4\sum_{q=0}^{\infty} a_{q-1}x^q - 16\sum_{n=0}^{\infty} a_n x^n = 0 \tag{13}$$

Finally, notice that the sum in the first term in (13) can have its lower index changed to $m = 0$ since when $m = 0$ we have $a_1(1)(0)x^0 = 0$ and therefore

$$\sum_{m=1}^{\infty} a_{m+1}(m+1)mx^m = \sum_{m=0}^{\infty} a_{m+1}(m+1)mx^m \tag{14}$$

Since indices are just labels we can rewrite everything in terms of n to get

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)nx^n - 4\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \\ 3\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + 4\sum_{n=0}^{\infty} a_{n-1}x^n - 16\sum_{n=0}^{\infty} a_n x^n = 0 \tag{15}$$

Combining the sums we have

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)nx^n - 4a_{n+2}(n+2)(n+1)x^n + 3a_{n+1}(n+1)x^n + 4a_{n-1}x^n - 16a_n x^n = 0 \tag{16}$$

which implies, since the coefficients of x^n for all n must be 0, that

$$a_{n+1}(n+1)n - 4a_{n+2}(n+2)(n+1) + 3a_{n+1}(n+1) + 4a_{n-1} - 16a_n = 0 \tag{17}$$

Noticing that $3(n+1) + (n+1)n = (n+1)(n+3)$ we can rewrite the coefficient of a_{n+1} to get

$$a_{n+1}(n+1)(n+3) - 4a_{n+2}(n+2)(n+1) + 4a_{n-1} - 16a_n = 0 \tag{18}$$

and we can then solve for a_{n+2} to get the recursion relation

$$a_{n+2} = \frac{(n+3)}{4(n+2)}a_{n+1} + \frac{1}{(n+2)(n+1)}a_{n-1} - \frac{4}{(n+2)(n+1)}a_n \tag{19}$$

The initial conditions tell us that $a_0 = 3$ and $a_1 = 0$. Using these values of a_0 and a_1 and the recursion relation (19) we can compute the first few terms of the series

$$\begin{aligned} a_0 &= 3 \\ a_1 &= 0 \\ a_2 &= -6 \\ a_3 &= -\frac{3}{2} \\ a_4 &= \frac{49}{32} \end{aligned}$$

and so on and so forth. Therefore, $y(x)$ is

$$\boxed{y(x) = 3 - 6x^2 - \frac{3}{2}x^3 + \frac{49}{32}x^4 + \frac{147}{320}x^5 - \frac{923}{7680}x^6 ... \qquad (20)}$$

What about the radius of convergence of this series? First, notice that the power series (10) that we used is an expansion around $x_0 = 0$. Then notice that the coefficients of $\frac{dy}{dx}$ and y in (9) can be expanded in Taylor series about the point $x_0 = 0$:

$$\frac{3}{x-4} \approx -\frac{3}{4} - \frac{3}{(x-4)^2}x + \frac{6}{(x-4)^3}\frac{1}{2!} + ... \tag{21}$$

and of course the Taylor series of 4 is simply 4.
Notice that $\frac{3}{x-4}$ has a singularity at $x = 4$. Therefore we might expect that a Taylor series expansion of $\frac{3}{x-4}$ about $x_0 = 0$ doesn't 'run into trouble' until $x = 4$ and therefore the radius of convergence of this series is $R = 4$. The Taylor series expansion for 4 has an infinite radius of convergence.
Finally, we might expect that the radius of convergence of our power series solution (20) is at least as great as the smallest radius of convergence of the coefficient expansions and therefore we might estimate that the radius of convergence is at least $R = 4$ for our solution (20).
We can see that $R = 4$ for (20) by using Theorem 4.2.2 of Greenberg. This theorem tells us that

$$R = \frac{1}{\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|} \tag{22}$$

for a power series. From (19) notice that

$$a_{n+2} \approx \frac{1}{4}a_{n+1} \tag{23}$$

as $n \to \infty$ which also means $a_{n+1} \approx \frac{1}{4}a_n$ as $n \to \infty$ and using the theorem we easily see that

$$\boxed{R = \frac{1}{\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|} = \frac{1}{\frac{1}{4}} = 4}$$

Problem 3: Deriving Fourier series represetnation

From: AM105 P-Set 5 2013

Derive the fourier represetnation for

$$f(x) = x^4 \tag{24}$$

defined on $0 \leq x \leq 1$.

Solution

$f(x) = x^4$ on the interval $0 \leq x \leq 1$ is not periodic. Therefore, how do we expand it in a fourier series? We will follow the treatment in Greenberg 17.4, namely, we expand $f(x) = x^4$ on the interval $0 \leq x \leq 1$ to the range $-\infty \leq x \leq +\infty$ to define an extended function f_{ext} that is periodic and has $f_{ext} = f(x)$ on $0 \leq x \leq 1$
We can extend $f(x)$ to $-\infty \leq x \leq +\infty$ in a variety of ways, namely we have the half-range cosine extension (HRC), half-range sine extension (HRS), quarter-range cosine extension(QRC) and the quarter-range sine extension (QRS). You should read section 17.4 to understand the difference between these various extensions and how they lead to the corresponding formulas for the fourier series.

In any event, for our function $f(x) = x^4$ let's expand it in a HRC (you could also use a HRS, QRC, or QRS and you would get different fourier coefficients) so that we have the corresponding fourier series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) \tag{25}$$
$$a_0 = \frac{1}{L}\int_0^L f(x)dx$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx$$

where L corresponds to the endpoint of the interval, for our problem $L = 1$.
Using (25) we have

$$a_0 = \frac{1}{1}\int_0^1 x^4 dx = \frac{x^5}{5}\Big|_0^1 = \frac{1}{5} \tag{26}$$

and using integration by parts to solve for the a_n we find that

$$a_n = \frac{2}{1}\int_0^1 x^4 \cos(\frac{n\pi x}{1})dx = \frac{8\cos(n\pi)}{(n\pi)^2} - \frac{48\cos(n\pi)}{(n\pi)^4} \tag{27}$$

thus one possible fourier series approximation to $f(x) = x^4$ is

$$\boxed{x^4 \approx \frac{1}{5} + \sum_{n=1}^{\infty}\left(\frac{8\cos(n\pi)}{(n\pi)^2} - \frac{48\cos(n\pi)}{(n\pi)^4}\right)\cos(n\pi x)}$$

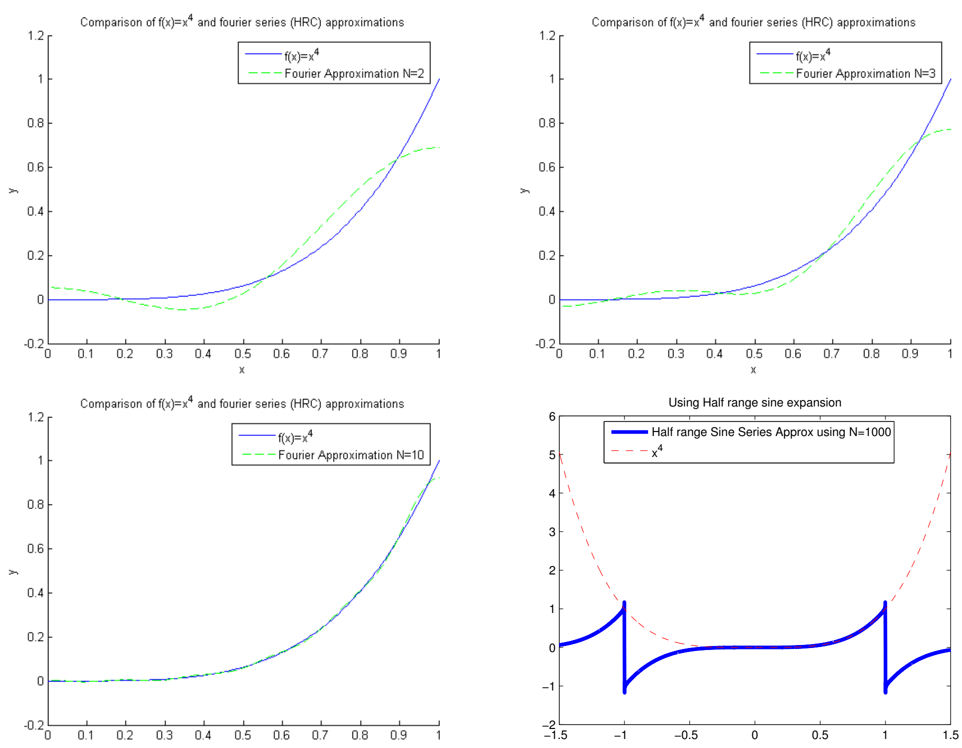

Figure 1: Figure showing different truncations of the series approximation. As you can see, the more n we use, the better the approximation to $f(x) = x^4$. Furthermore, the right bottom graph shows what the approximation would look like if we used a half range sine expansion instead of the half range cosine expansion.

Matlab file for half range graph

```
clc
close all
clear all

bn = @(n) 2*(4*(pi*n*(pi^2*n^2-6)*sin(n*pi)+6)-(pi^4*n^4-12*pi^2*n^2+24)*cos(pi*n))/(pi^5*n^5);
x=-1.5:0.001:1.5;
f=zeros(1,length(x));
fanaly=x.^4;
for n=1:1000
f=f+bn(n)*sin(n*pi.*x);
end

plot(x,f,'linewidth',3); hold on
plot(x,fanaly,'--r')
legend('Half range Sine Series Approx using N=1000','x^4')
title('Using Half range sine expansion')
```