APMTH-105 Notes Section #6

Matheus C. Fernandes

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Goals for the week

1. Learn about the convergence of Fourier series and the Gibbs phenomenon.
2. Learn how to solve a Sturm-Liouville problem.

Problem 1: Fourier Series Convergence/ Gibbs phenomenon

From: Practice Problem Set

Consider the function $f(x) = 1$ for $0 \leq x \leq 1$ and $f(x) = -1$ for $-1 \leq x \leq 0$.

a Compute the Fourier series expansion of the function.

b Explain what the Gibbs phenomenon is. Plot partial sums of your expansion and compare your calculation of the Gibbs phenomenon to our discussion of it in class. Does the overshoot have the same value, even though the function we are expanding is so different?

Solution: a

First, we note that because the step function being expanded is odd, we will only require $\sin$ terms. To determine the coefficients b_n, we need to calculate:

$$\begin{aligned} b_n &= \frac{\langle f(x), \sin(n\pi x/L)\rangle}{\langle \sin(n\pi x/L), \sin(n\pi x/L)\rangle} \\ &= \frac{\int_{-1}^{1} f(x)\sin(n\pi x)}{\int_{-1}^{1}\sin^2(n\pi x)} \\ &= \int_{-1}^{1} f(x)\sin(n\pi x) \end{aligned}$$

Plugging in for $f(x)$, we find:

$$\begin{aligned} b_n &= -\int_{-1}^{0}\sin(n\pi x) + \int_{0}^{1}\sin(n\pi x) \\ &= \frac{1}{n\pi}\left[\cos(n\pi x)\Big|_{x=-1}^{x=0} - \cos(n\pi x)\Big|_{x=0}^{x=1}\right] \end{aligned}$$

For even n, we see that $b_n = 0$. For odd n, we see that $b_n = \frac{4}{n\pi}$.
Therefore, our Fourier expansion is:

$$\boxed{f(x) = \frac{4}{\pi}\sum_{i=1,3,5...}\frac{1}{n}\sin(n\pi x)}$$

Solution: b

Gibbs phenomenon describes the failure of Fourier expansions around jump discontinuities of piecewise continuous functions. Fourier series overshoot jump discontinuities by a finite amount that persists in the limit of infinite Fourier terms.
Fig. 1 verifies the convergence of our Fourier Series expansion with 10 terms (red), 50 terms (black), and 100 terms (green).

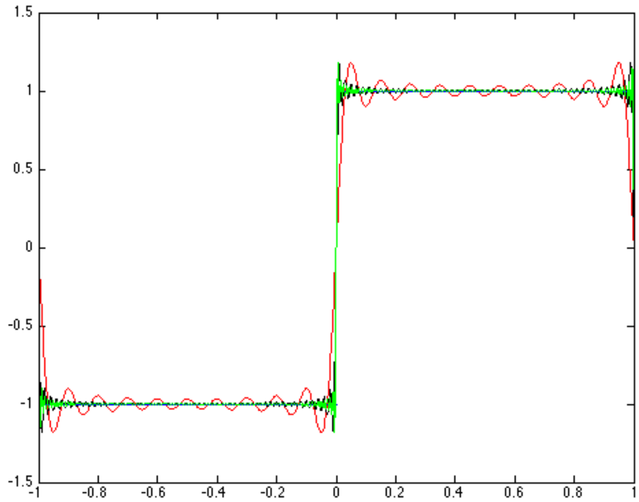

Figure 1: Convergence of Fourier series for step function

Zooming into our figure for the Fourier expansion, we see that the series converges approximately to the value of 1.18 for a total overshoot of approximately

$$\frac{2.18-2}{2} = .09 = 9\%$$

This is very close to what we found in class (see lecture 17).

Idea 1: Regular Sturm-Liouvilel Problem

A regular Sturm-Liouville problem is a boundary value problem on a closed finite interval $[a, b]$ of the form

$$[p(x)y']' + [q(x) + \lambda w(x)]y = 0, \qquad a < x < b, \tag{1}$$

$$\begin{array}{l} \text{(a)} \\ \text{(b)} \end{array} \begin{cases} c_1 y(a) + c_2 y'(a) = 0, \\ d_1 y(b) + d_2 y'(b) = 0. \end{cases} \tag{2}$$

where at least one of c_1 and c_2 and at least one of d_1 and d_2 are nonzero, and λ is a parameter. Equation (1) is said to be in **Sturm-Liouville form**.

Problem 2: Sturm-Liouville Problem

Determine the eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$y'' + \lambda y = 0 \tag{3}$$
$$y(0) = 0, \quad y'(\pi) = 0 \tag{4}$$

Solution:

Before we proceed with the solution, we can use out knoledge of Fourier series to guess a family of orthogonal functions that satisfy the Sturm-Liouville problem: $y_k(x) = \sin \frac{2k+1}{2} x, k = 0, 1, 2, \ldots$. It is straightforward to check the validity of our guess. Let us instead proceed to derive these solutions.

Using the anzats: $y = e^{rx}$ we get:

$$r^2 e^{rx} + \lambda e^{rx} = 0 \Rightarrow r^2 = -\lambda \tag{5}$$

So, for this example we consider three possible cases.

Case I: $\lambda < 0$. Let us write $\lambda = -\alpha^2$ where $\alpha > 0$. Then the equation becomes $y'' = -\alpha^2 y = 0$, and its general solution is $y = c_1 \sinh \alpha x + c_2 \cosh \alpha x$. We need $y(0) = 0$, so substituting into the general solution gives $c_2 = 0$. Now using the condition $y'(\pi) = 0$, we get $0 = c_1 \alpha \cosh \alpha \pi$, and since $\cosh x \neq 0$ for all x, we infer that $c_1 = 0$. Thus there are no nonzero solution in this case.

Case 2: $\lambda = 0$. Here the general solution of the differential equation is $y = c_1 x + c_2$, and as in Case 1 the boundary conditions force c_1 and c_2 to be 0. Thus again there is no nonzero solution.

Case 3: $\lambda > 0$. In this case we can write $\lambda = \alpha^2$ with $\alpha > 0$, and so the equation becomes $y'' + \alpha^2 y = 0$. The general solution is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$, From $y(0) = 0$ we get $0 = c_1 \cos 0 + c_2 \sin 0$ or $0 = c_1$. Thus $y = c_2 \sin \alpha x$. Now we substitute the other boundary condition to get $0 = c_2 \alpha \cos \alpha \pi$. Since we are seeking nonzero solutions, we take $c_2 \neq 0$. Thus we must have $\cos \alpha \pi = 0$, and hence $\alpha = \frac{2k+1}{2}$. Since $\lambda = \alpha^2$, the problem has eigenvalues

$$\lambda_k = \left(\frac{2k+1}{2}\right)^2,$$

and corresponding eigenfunctions

$$y_k = \sin \frac{2k+1}{2}, k = 0, 1, 2, \ldots$$

Problem 3: Sturm-Liouville Problem

Determine the eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$y'' + \lambda y = 0 \tag{6}$$
$$y(0) + y'(0) = 0, \quad y(1) + y'(1) = 0 \tag{7}$$

Solution:

Using the anzats: $y = e^{rx}$ we get:

$$r^2e^{rx} + \lambda e^{rx} = 0 \Rightarrow r^2 = -\lambda \tag{8}$$

So, for this example we consider three possible cases.

Case I If $\lambda = 0$, the general solution of the differential equation is $X = ax + b$. We then check that the only way to satisfy the boundary conditions is to take $a = b = 0$. Thus $\lambda = 0$ is not an eigenvalue since no nontrivial solution exists.

Case 2 If $\lambda = -\alpha^2 < 0$, then the general solution of the differential equation is $X = c_1 \cosh \alpha x + c_2 \sinh \alpha x$. We have that $X' = c_1\alpha \sinh \alpha x + c_2\alpha \cosh \alpha x$. In order to have nonzero solutions, we suppose throughout the solution that c_1 or c_2 is nonzero. The first boundary condition implies

$$c_1 + \alpha c_2 = 0 \quad c_1 = -\alpha c_2$$

Hence both c_1 and c_2 are nonzero. The second boundaru condition implies that

$$c_1(\cosh \alpha + \alpha \sinh \alpha) + c_2(\sinh \alpha + \alpha \cosh \alpha) = 0$$

Using $c_1 = -\alpha c_2$, we obtain

$$\begin{aligned} -\alpha c_2(\cosh \alpha + \alpha \sinh \alpha) + c_2(\sinh \alpha - \alpha \cosh \alpha) &= 0 \text{ (divide by} c_2 \neq 0) \\ \sinh \alpha(1 - \alpha^2) &= 0 \\ \sinh \alpha = 0 \text{ or } 1 - \alpha^2 &= 0 \end{aligned}$$

Since $\alpha \neq 0$, it follows that $\sinh \alpha \neq 0$ and this implies that $1 - \alpha^2 = 0$ or $\alpha = \pm 1$. We take $\alpha = 1$,because the value -1 does not yield any new eigenfunctions. For $\alpha = 1$, the corresponding solution is

$$X = c_1 \cosh x + c_2 \sinh x = -c_2 \cosh x + c_2 \sinh x,$$

because $c_2 = -\alpha c_2 = -c_2$. So in this case we have one negative eigenvalue $\lambda = -\alpha^2 = -1$ with corresponding eigenfunction $X = \cosh x - \sinh x$.

Case 3 If $\lambda = \alpha^2 > 0$, then the general solution of the differential equation is

$$X = c_1 \cos \alpha x + c_2 \sin \alpha x.$$

We have $X' = -c_1\alpha \sin \alpha x + c_2\alpha \cos \alpha x$. In order to have nonzero solutions, one of the coefficients c_1 or c_2 must be $\neq 0$. Using the boundary conditions, we obtain

$$\begin{aligned} c_1 + \alpha c_2 &= 0 \\ c_1(\cos \alpha - \alpha \sin \alpha) + c_2(\sin \alpha + \alpha \sin \alpha) &= 0 \end{aligned}$$

The first equation implies that $c_1 = -\alpha c_2$ and so both c_1 and c_2 are $\neq 0$. From the second equation, we obtain

$$\begin{aligned} -\alpha c_2(\cos \alpha - \alpha \sin \alpha) + c_2(\sin \alpha + \alpha \cos \alpha) &= 0 \\ -\alpha(\cos \alpha - \alpha \sin \alpha) + (\sin \alpha + \alpha \cos \alpha) &= 0 \\ \sin \alpha(\alpha^2 + 1) &= 0 \end{aligned}$$

Since $\alpha^2 + 1 \neq 0$, then $\alpha = 0$, and so $\alpha = \pi n$, where $n = 1, 2, \ldots$. Thus the eigenvalues are

$$\lambda_n = (n\pi)^2$$

with corresponding eigenfunctions

$$y_n = -n\pi \cos n\pi x + \sin n\pi x, \qquad n = 1, 2, \ldots.$$